A note on the T-uniqueness of graph *

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Abstract

A graph G is called *T-unique* if any other graph having the same Tutte polynomial as G is isomorphic to G. Recently, there has been much interest in determining T-unique graphs and matroids. Several families of graphs have been proven to be T-unique [4–8]. In this paper, we study the T-uniqueness of a new class of graphs constructed from two half wheels by sharing one triangle.

Keywords: Tutte polynomial; T-unique; double half-wheel.

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1 Introduction

Let G be a graph with vertex set V and edge set E. We assume that G has no isolated vertices, but loops and multiple edges are allowed. The *rank* of a subset S of E is the number of edges in a spanning forest of the subgraph induced by S in G, i.e. $r(S) = |V| - k(G|S)$, where $k(G|S)$ denotes the number of components of spanning subgraph induced by S in G.

The *Tutte polynomial* of G is defined as

$$T(G; x, y) = \sum_{S \subseteq E} (x-1)^{r(E)-r(S)}(y-1)^{|S|-r(S)}.$$

Two graphs G_1 and G_2 are called *T-equivalent*, if $T(G_1; x, y) = T(G_2; x, y)$. A graph G is called *T-unique* provided all graphs T-equivalent to G are graphs isomorphic to G [7].

In [7], de Mier and Noy introduced a new polynomial, the *rank-size generating polynomial*, as follows

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$$F(G;x,y) = \sum_{S\subseteq E} x^{r(S)} y^{|S|}.$$

It is not hard to see that the coefficient of each monomial $x^{r(S)}y^{|S|}$ in $F(G;x,y)$ can be derived from $T(G;x,y)$, and vice versa. Therefore, $F(G;x,y)$ and $T(G;x,y)$ contain exactly the same information about G. It is relatively easier to extract information from $F(G;x,y)$ than from $T(G;x,y)$, so it becomes a useful tool to mine invariant properties for $T(G;x,y)$ and thus to prove the T-uniqueness property for many families of graphs.

The coefficients of the Tutte polynomial of a graph contain a lot of information about the graph, such as graphic parameters shown in the next two theorems. We will use these results frequently in our proofs later.

Theorem 1.1 (de Mier and Noy [7], Theorem 2.4) *Let $G=(V,E)$ be a 2-connected graph, then the following graphic parameters of G are determined by its Tutte polynomial:*

i) The number of vertices and the number of edges;

ii) For every k, the number of edges with multiplicity k. In particular, whether G is a simple graph or not;

iii) The number of shortest cycles;

iv) The edge-connectivity $\lambda(G)$;

v) If G is simple, the number of cliques of each size. In particular, the clique-number $\omega(G)$;

vi) If G is simple, the number of cycles of lengths three, four and five. For the cycles of length four, it is also possible to know how many of them have exactly one chord. □

Let $n(G) = |E(G)| - r(G)$. The following well-known result is a phrased version for graphic matroids (see [7, Theorem 2.2]). A slightly more general result for matroids can be found, for example, in [2, Example 6.2.17].

Theorem 1.2 *If both of $r(G)$ and $n(G)$ are positive, then the number of blocks of G (that is, 2-connected components) is $min\{i \mid b_{i0} \neq 0\}$. Otherwise, G has $|E(G)|$ blocks. In particular, if G is 2-connected and H is T-equivalent to G, then H is also 2-connected.* □

A *bond* of a graph is a minimal edge-cut. In [3], it is proved that for a 2-connected graph G with $\lambda(G) \geq 3$, the number of the minimum bonds is determined by its flow polynomial. As the flow polynomial of a graph is an evaluation of its Tutte polynomial (see [1]), the number of minimum bonds of a graph can also be determined by its Tutte polynomial. We will use this fact in Section 2.

Several classes of graphs have been proven to be T-unique. In [7], de Mier and Noy proved that wheels W_n, square of cycles C_n^2, complete multipartite graphs $K_{p_1,p_2,\cdots,p_r}$, ladders L_n, Möbius ladders M_n and hypercubes Q_n are T-unique. In [5], it is proved that generalized Petersen graph $P(m,2)$ is T-unique, and in the same paper, Kuhl also proved that the line graph of $P(m,2)$ is T-unique. In [8], de Mier and Noy consider the T-uniqueness of line graphs and proved that the line graph of complete graphs K_n, complete bipartite graphs $K_{p,q}$ and regular complete t-partite graphs $K(p,t)$ ($t \geq 2$) are T-unique. In [6], Márquez, de Mier, Noy and Revuelta proved that the locally grid graphs are T-unique. In [4], it was proved that the twisted wheel is T-unique. In this paper, we intend to extend the body of knowledge on T-unique classes of graphs. The same technique used in [4] is adopted again to prove the T-uniqueness of a new class of graphs.

The *double half-wheel* $W_h(k_1, k_2)$ (see Fig. 1(b)) is a graph obtained from the graph G (see Fig. 1(a)) with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ by subdividing the edges v_1v_5 and v_4v_5 with $k_1 - 2$ and $k_2 - 1$ vertices, and then joining each of the new vertices on v_1v_5 and v_4v_5 to v_2 and v_3, respectively. Clearly, $W_h(k_1, k_2)$ has $k_1 + k_2$ triangles. The edge $e_0 = v_{1,1}v_{2,k_2}$ is called the *strap edge*.

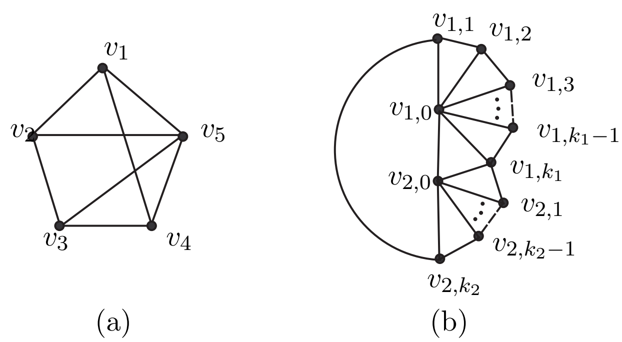


Figure 1: The graph of $W_h(k_1, k_2)$

Remark: In the rest of the paper, we assume that $k_1 \geq k_2 \geq 2$, $k_1 + k_2 \geq 5$ and $n = k_1 + k_2 + 2$. Then in $W_h(k_1, k_2)$, there are n vertices, $2n-2$ edges, $n-2$ triangles, $n-2$ C_4's, $n-3$ C_4^+'s and all vertices and all edges except the strap edge are contained in triangles, where C_4^+ denotes a C_4 with one chord. In this paper, we use $K_{q,2}^+$ to denote the graph constructed from q triangles sharing a single edge. Clearly, $W_h(k_1, k_2)$ contains no $K_{3,2}^+$ as subgraph.

From Theorem 1.1 *ii*), we see that any graph which is T-equivalent to a simple graph is also simple. Although the Tutte polynomials are defined on all graphs, in this paper we restrict our attention to the T-uniqueness of simple graphs only. From now on, all graphs considered are simple.

In [4], we introduced a new concept similar to the line-graph called the *triangle-graph* $TR(G)$. In $TR(G)$, each vertex corresponds to a triangle in G and two vertices are adjacent if and only if the corresponding triangles share an edge in G. Clearly, the graph $TR(G)$ is well-defined and simple. It is a good tool for reconstructing several classes of graphs such as the twisted wheel in [4]. It is easy to see that the double half-wheel can be viewed as a variation of the twisted wheel. Therefore, we use this technique again to prove $W_h(k_1, k_2)$ to be T-unique.

Here are some useful notions included in [4]. For a graph G, the subgraph induced by all edges contained in some triangle of G is called the *triangle-induced subgraph* of G, denoted by $\hat{G}$. If the triangle-graph $TR(G)$ is a tree of n vertices, then the graph G is called a *triangular n-tree graph* and the set of all such graphs on n vertices is denoted by Γ^n. If $TR(G)$ is a path of order n, then G is called a *triangular n-path graph* and the set of all such graphs is denoted by $\wp^n$. If $TR(G)$ is a cycle of length n, then G is called a *triangular n-cycle graph* and the set of all such graphs is denoted by $\mathcal{C}^n$. If $TR(G)$ is a forest of n vertices with r components, then G is called a *triangular n-forest with r components graph* and the set of all such graphs is denoted by $\mathcal{F}^n_r$.

For convenience, we use T^n, P^n, C^n, and F^n_r to denote any graph in Γ^n, $\wp^n$, $\mathcal{C}^n$ and $\mathcal{F}^n_r$, respectively. For these families of graphs, we have the following result which has been included in [4].

Proposition 1.3 *i) For any $T^n \in \Gamma^n$, $|V(\hat{T}^n)| \leq n+2$ and $|E(\hat{T}^n)| = 2n+1$;*

ii) For any $C^n \in \mathcal{C}^n$, $n \geq 4$, $|V(\hat{C}^n)| \leq n+1$ and $|E(\hat{C}^n)| = 2n$. □

Suppose that G is a connected graph in $\wp^n$ such that $V(G) = V(\hat{G})$, $E(G) = E(\hat{G})$ and $|V(\hat{G})| = n+2$. Then G is called a *maximal triangular n-path*, denoted by P^n_{max}. The set of all such graphs will be denoted by $\wp^n_{max}$. Clearly, the maximal triangle-induced n-path P^n_{max} has exactly two vertices of degree 2. The following result has been proved in [4].

Lemma 1.4 *Suppose that G is a graph satisfying the following conditions*

i) G is 3-edge-connected,

ii) $\hat{G} \in \wp^{n-2}_{max}$, where $n \geq 8$, and

iii) $E(G) = E(\hat{G}) \cup \{e\}$, where e is an edge joining the two 2-degree vertices of $\hat{G}$.

Then every 3-element bond of G is trivial. □

It is clear that in a tree, the number of paths of length two (or 2-paths) is equal to the number of edges of its line graph. The following result which is also included in [4] gives a new way to count the number of 2-paths, which will be useful later in this paper.

Lemma 1.5 *Let T be a tree with n vertices and the maximum degree $\Delta(T)$. Let m_i be the number of vertices of degree i in T. Then the number of 2-paths is $n - 2 + \sum_{i=3}^{\Delta(T)} m_i \binom{i-1}{2}$, and hence T has $n-2$ 2-paths if and only if T is an n-path.* □

2 Proof of the main result

In this section, we prove our main result.

Theorem 2.1 *If $k_1 \geq k_2 \geq 2$ and $k_1 + k_2 \geq 5$, the double half-wheel $W_h(k_1, k_2)$ is T-unique.*

A *k-fan* F_k is a maximal triangular k-path in which the k triangles share a common vertex v (shown in Fig. 2). The edges v_0v_1 and $v_{k-1}v_k$ are called the *brim non-spoke-edges* of F_k.

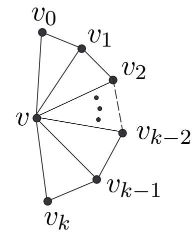


Figure 2: k-fan F_k

Following [7], we use $[x^i y^j]F(G;x,y)$ to denote the coefficient of the term $x^i y^j$ in $F(G;x,y)$.

Proposition 2.2 *Let $A \subseteq E(W_h(k_1,k_2))$. The following statements hold:*

i) $[x^k y^{2k+i}]F(W_h(k_1,k_2);x,y) = 0$, for $k \le n-2$ and $i \ge 0$;

ii) if the strap edge $e_0 \notin A$, then the graphs induced by the edge set A contributing to the coefficient of $x^k y^{2k-1}$ are the subgraphs of $W_h(k_1,k_2)$ which is isomorphic to graphs in $\wp^{k-1}_{max}$;

iii) if the strap edge $e_0 \in A$, then the edge sets contributing to the coefficient of $x^k y^{2k-1}$ are described in (1.1)-(1.8), (2.1)-(2.3) and (3.1)-(3.3).

Proof. Let $A = A_0 \cup A_0' \cup A_0''$, where $A_0 = \{e_0\}$ or $\varnothing$, A_0' is the set of edges in $E(W_h(k_1,k_2)) - A_0$ contained in some cycle of $G[A - A_0]$ and A_0'' is the set of edges in $E(W_h(k_1,k_2)) - A_0$ contained in no cycles of $G[A - A_0]$.

For all of the indices of these vertices in $W_h(k_1,k_2)$, we define an ordering: $(1,1) < (1,2) < \cdots < (1,k_1-1) < (1,0) < (1,k_1) < (2,1) < (2,2) < \cdots < (2,k_2-1) < (2,k_2) < (2,0)$. Let $\mathscr{C}_1$, $\mathscr{C}_2$, $\cdots$, $\mathscr{C}_m$ be the chordless cycles of $G[A_0']$ with the order of these cycles is given by the minimum indice of vertices of each cycle. Let c_1, c_2, $\cdots$, c_m denote the size of the cycles $\mathscr{C}_1$, $\mathscr{C}_2$, $\cdots$, and $\mathscr{C}_m$, respectively. Since $G[A_0']$ is a subgraph of $W_h(k_1,k_2)$, every pair of chordless cycles share at most one edge.

Let $f = |A_0''|$. Define θ_i, ψ and ψ' as follows.

$$\theta_i = \begin{cases} 1, & \text{if } i = 1; \\ 2, & i \ge 2 \text{ and } \mathscr{C}_i \text{ and } \mathscr{C}_{i-1} \text{ share one edge;} \\ 1, & i \ge 2 \text{ and } \mathscr{C}_i \text{ and } \mathscr{C}_{i-1} \text{ share no edge.} \end{cases}$$

$$\psi = \begin{cases} 0, & \text{if } e_0 \notin A; \\ 1, & \text{otherwise.} \end{cases}$$

$$\psi' = \begin{cases} 0, & \text{if } e_0 \notin A \text{ or } e_0 \text{ contained in a cycle of } G[A]; \\ 1, & \text{otherwise.} \end{cases}$$

Then we have the following equations:

$$|A| = \sum_{i=1}^{m}(c_i - \theta_i + 1) + f + \psi = \sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + m + f + \psi, \quad (1)$$

$$r(A) = \sum_{i=1}^{m}(c_i - \theta_i) + f + \psi' = \sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + \psi'. \quad (2)$$

Since $c_i \geq 3$, $\theta_i \leq 2$ $(2 \leq i \leq m)$, $\theta_1 = 1$ and $f \geq 0$, we deduce that

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f \geq m + 1. \quad (3)$$

By Equations (1) and (2) and Inequality (3), we have $|A| \leq 2r(A)$. Evidently, if $i > 0$, then $[x^k y^{2k+i}]F(W_h(k_1, k_2); x, y) = 0$, for $k \leq n - 1$.

If $|A| = 2r(A)$, from Equations (1) and (2), we have

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + 2\psi' = m + \psi. \quad (4)$$

If $\psi = 0$, then $\psi' = 0$ by definition. From (4), we have $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f = m$, a contradiction to Inequality (3).

Thus, $\psi = 1$, and by Inequality (3) and Equation (4), $\psi' = 0$. Thus, equality holds in (3), that is $c_i = 3$ $(1 \leq i \leq m)$, $\theta_i = 2$ $(2 \leq i \leq m)$ and $f = 0$, therefore $A = E(W_h(k_1, k_2))$. That is, in $F(W_h(k_1, k_2); x, y)$ the only edge subset contributing to the coefficient of $x^k y^{2k}$ is $E(W_h(k_1, k_2))$, and the only term is $x^{n-1}y^{2n-2}$ with coefficient 1. Thus *i*) holds.

Next consider the edge subset A contributing to the coefficient of $x^k y^{2k-1}$, $1 \leq k \leq n-1$, i.e. $|A| = 2r(A) - 1$. By Equations (1) and (2), we have

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f + 2\psi' - 1 = m + \psi. \quad (5)$$

If $e_0 \notin A$, then $\psi = 0$ and $\psi' = 0$ by definition. Then by Equation (5), we have $\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f - 1 = m$.

Thus equality holds in (3), that is $c_i = 3$ $(1 \leq i \leq m)$, $\theta_i = 2$ $(2 \leq i \leq m)$, $f = 0$ and $m + 1 = k$. Therefore, if $e_0 \notin A$, then $G[A]$ is a subgraph of $W_h(k_1, k_2)$ isomorphic to a graph in $\wp_{max}^{k-1}$, as required. Thus *ii*) holds.

If $e_0 \in A$, then $\psi = 1$. Thus by (5), we have

$$\sum_{i=1}^{m} c_i - \sum_{i=1}^{m} \theta_i + f - 1 + 2\psi' = m + 1. \quad (6)$$

Inequality (3) implies that $\psi' = 0$, i.e., $e_0 \in A$ and e_0 is contained in a cycle of $G[A]$.

To satisfy (6), there are three possible cases.

In the following, we say two chordless cycles in $W_h(k_1,k_2)$ are *consecutive* if they share one edge. A sequence of chordless cycles $\mathscr{C}_i$, $1\le i\le m$, are consecutive if $\mathscr{C}_i$ and $\mathscr{C}_{i+1}$ share one edge.

Case 1. $c_i=3$ $(1\le i\le m)$, $\theta_i=2$ $(2\le i\le m)$ and $f=1$.

In this case, the chordless cycles $\mathscr{C}_1$, $\mathscr{C}_2$, $\cdots$, $\mathscr{C}_m$ are consecutive and with length 3. In addition, since $f=1$, e_0 is contained in a cycle of $G[A]$. Thus, A is one of the following sets.

(1.1) $A=E-\cup_{i=1}^{k_2-1}v_{2,0}v_{2,i}-\cup_{j=1}^{k_2-1}v_{2,j}v_{2,j+1}-v_{1,k_1}v_{2,1}$;

(1.2) $A=E-\cup_{i=2}^{k_1-1}v_{1,0}v_{1,i}-\cup_{j=1}^{k_1-1}v_{1,j}v_{1,j+1}$;

(1.3) $A=E-\cup_{i=t}^{k_2-1}v_{2,0}v_{2,i}-\cup_{j=t}^{k_2-1}v_{2,j}v_{2,j+1},\ t\ge 2$;

(1.4) $A=E-\cup_{i=2}^{t}v_{1,0}v_{1,i}-\cup_{j=1}^{t}v_{1,j}v_{1,j+1}$, $t\ge 2$;

(1.5) $A=E-v_{2,0}v_{2,k_2}$;

(1.6) $A=E-v_{1,0}v_{1,1}$;

(1.7) $A=E-v_{2,k_2-1}v_{2,k_2}$;

(1.8) $A=E-v_{1,1}v_{1,2}$.

Case 2. There exists t, $1\le t\le m$, such that $c_t=4$, and for all $i\ne t$, $c_i=3$, $\theta_i=2$ $(2\le i\le m)$ and $f=0$.

In this case, the chordless cycles $\mathscr{C}_i$ are consecutive and with length 3 except one. Thus, A is one of the following sets.

(2.1) $A=E-v_{1,0}v_{1,i}$, $2\le i\le k_1$;

(2.2) $A=E-v_{2,0}v_{2,i}$, $2\le i\le k_2-1$;

(2.3) $A=E-v_{2,0}v_{1,k_1}$.

Case 3. There exists t, $2\le t\le m$, such that $\theta_t=1$ and for all $i\ne t$, $\theta_i=2$, $c_i=3$ $(1\le i\le m)$ and $f=0$.

In this case, the chordless cycles $\mathscr{C}_i$, $1\le i\le m$ are not consecutive and have exactly one "gap". Thus, A is one of the following sets.

(3.1) $A=E-\cup_{i=r}^{s}v_{2,0}v_{2,i}-\cup_{j=r}^{s+1}v_{2,j-1}v_{2,j}$, $2\le r\le s\le k_2-2$;

(3.2) $A=E-\cup_{i=1}^{t}v_{2,0}v_{2,i}-\cup_{j=1}^{t}v_{2,j}v_{2,j+1}-v_{1,k_1}v_{2,1}, t\le k_2-2$;

(3.3) $A=E-\cup_{i=r}^{s}v_{1,0}v_{1,i}-\cup_{j=r}^{s+1}v_{1,j-1}v_{1,j}$, $3\le r\le s\le k_1-1$.

This completes the proof of the proposition. □

Remark: All the subgraphs contributing to the coefficient of x^ky^{2k-1} are listed in Proposition 2.2 *ii*) and *iii*). The rank of all the subgraphs in (1.1)-(1.8), (2.1)-(2.3) and (3.1)-(3.3) are at least k_2+2. Therefore, in $W_h(k_1,k_2)$, if $k\le k_2+1$, then all graphs induced by A contributing to the coefficient of x^ky^{2k-1} are the subgraphs of $W_h(k_1,k_2)$ isomorphic to graphs in $\wp_{max}^{k-1}$ and the total number of subgraphs is $n-k$. If $k\ge k_2+2$, then all graphs induced by A contributing to the coefficient of x^ky^{2k-1} are the subgraphs isomorphic to graphs in $\wp_{max}^{k-1}$ (subgraphs in Proposition 2.2 *ii*)) and the subgraphs mentioned in Proposition 2.2

iii), and there are at least $n-k+1$ such graphs in total.

Assume that a graph G is T-equivalent to $W_h(k_1,k_2)$. Our proof to show that G is isomorphic to $W_h(k_1,k_2)$ includes the following steps: 1) we will first show that $TR(G)$ is acyclic, 2) we will next show that, in fact, $TR(G)$ is a path and 3) we finally show that $TR(G)$ is a path of length $n-2$. Once these claims are confirmed, the main result follows easily.

Lemma 2.3 *If G is a graph T-equivalent to $W_h(k_1,k_2)$ and G contains no $K_{q,2}^+$ as a subgraph, $q \geq 4$, then $TR(G)$ contains no cycle of length p, $4 \leq p \leq n-2$.*

Proof. By way of contradiction, let $C_p = v_1v_2v_3\cdots v_pv_1$ be a shortest cycle in $TR(G)$ with the length p, $4 \leq p \leq n-2$.

Since G is a graph T-equivalent to $W_h(k_1,k_2)$, directly from Proposition 1.3 and Proposition 2.2, we see that G contains no C^p as a subgraph ($C^p \in \mathcal{C}^p$, $4 \leq p \leq n-2$), hence C_p has at least one chord v_1v_s. Let $S_1 = v_1v_2\cdots v_sv_1$ and $S_2 = v_1v_p\cdots v_sv_1$. Clearly, both S_1 and S_2 have lengths at least 3. Without loss of generality, we assume that $|S_1| \geq |S_2| \geq 3$. Since C_p is a shortest cycle with length p, $4 \leq p \leq n-2$, in $TR(G)$, we have $|S_1| = |S_2| = 3$. Then $TR(G)$ contains C_4^+ as a subgraph. It easy to see that G contains either $K_{4,2}^+$ or K_4 as a subgraph. In either case we have a contradiction since G contains the same number of K_4's as $W_h(k_1,k_2)$. □

Proof of Theorem 2.1: It was mentioned in Section 1 that the number of minimum bonds of graph G is determined by its Tutte polynomial. In addition, using Theorems 1.1 and 1.2 and the remark after the definition of double half-wheel $W_h(k_1,k_2)$, we know that if G is a graph T-equivalent to $W_h(k_1,k_2)$, then we have the following facts:

- G is simple and 2-connected;
- $|V(G)| = n$ and $|E(G)| = 2n-2$;
- the number of K_3's and K_4's are $n-2$ and 0, respectively;
- the number of C_4's and C_4^+'s are $n-2$ and $n-3$, respectively;
- G is 3-edge-connected, and the number of 3-element bonds is $n-3$.

Claim. There is no edge in $E(G)$ contained in more than two triangles.

Let τ_i be the number of edges of G contained in exactly i triangles, $i \geq 0$. Counting the total number of the times of edges appearing in the triangles of G, we conclude that the total number of the times of the edges appearing in at least three triangles is $\sum_{i=3}^{n-2} i\tau_i$; the total number of the times of the edges appearing in exactly two triangles is $2(n-3-\sum_{i=3}^{n-2}\tau_i\binom{i}{2})$, and the total number of the times of the edges appearing in at most one triangle is $2n-2-\sum_{i=3}^{n-2}\tau_i-(n-3-\sum_{i=3}^{n-2}\tau_i\binom{i}{2})$. Then, we have

$$3(n-2) \leq \sum_{i=3}^{n-2} i\tau_i + 2(n-3-\sum_{i=3}^{n-2}\tau_i\binom{i}{2}) + [2n-2-\sum_{i=3}^{n-2}\tau_i - (n-3-\sum_{i=3}^{n-2}\tau_i\binom{i}{2})] \quad (7)$$

That is, $\sum_{i=3}^{n-2}[\binom{i}{2}-(i-1)]\tau_i \leq 1$. Therefore $\tau_3 \leq 1$ and $\tau_i = 0$, for $i \geq 4$.

Thus, G contains no $K_{q,2}^{+}$ as a subgraph, where $q \geq 4$ and the number of $K_{3,2}^{+}$'s in G is $\tau_3 \leq 1$.

Considering the subgraphs which contribute to the coefficient of x^4y^7, there are four possible subgraphs:

a) $K_{3,2}^{+}$;

b) 3-fan;

c) a complete subgraph K_4 with an extra edge;

d) $K_{3,2}$ with an extra edge joining two vertices in the partite set of order three.

Since by Proposition 1.3 and Proposition 2.2, G contains no C^p, $4 \leq p \leq n-2$ or K_4. And every C_4 contains a chord except one, the subgraphs in *c*) and *d*) are impossible. Thus the possible subgraphs can only be $K_{3,2}^{+}$ and 3-fan. Let c be the number of subgraphs of G isomorphic to 3-fan. Then $\tau_3 + c = [x^4y^7]F(G;x,y) = [x^4y^7]F(W_h(k_1,k_2);x,y) = n-4$.

Next, we show that $\tau_3 = 0$.

Otherwise, $\tau_3 = 1$. Then G contains exactly one $K_{3,2}^{+}$ as a subgraph, denote it by G_0 (see Fig. 3(a)) and $c = n-5$. In this case, $TR(G)$ contains exactly one cycle and its length is 3 by Lemma 2.3.

Since $\tau_i = 0$, for $i \geq 4$ and $\tau_3 = 1$, the equality holds in (7), therefore every edge of G is contained in some triangle. Moreover, every vertex of G is contained in some triangle. In G_0, let $e_0 = uu'$, $e_i = uu_i$ and $e'_i = u'u_i$, $i = 1,2,3$ and denote the triangles uu_iu' by t_i. Let r be the number of triangles of $G - e_0$ containing edges e_i or e'_i, $i = 1,2,3$, then $0 \leq r \leq 6$.

Recall that a C_4^{+} in G is an edge of $TR(G)$ and a 3-fan is a 2-path of $TR(G)$. If $r = 6$, let $H_1 = G - e_0$, then there are $n-5$ triangles, $n-12$ C_4^{+}'s and at most $n-20$ 3-fans in H_1 and $TR(H_1)$ is acyclic. Thus $TR(H_1)$ is a forest with $n-5$ vertices and 7 components and by Lemma 1.5, there are at least $n-19$ 2-paths in $TR(H_1)$, a contradiction. So we assume $0 \leq r \leq 5$.

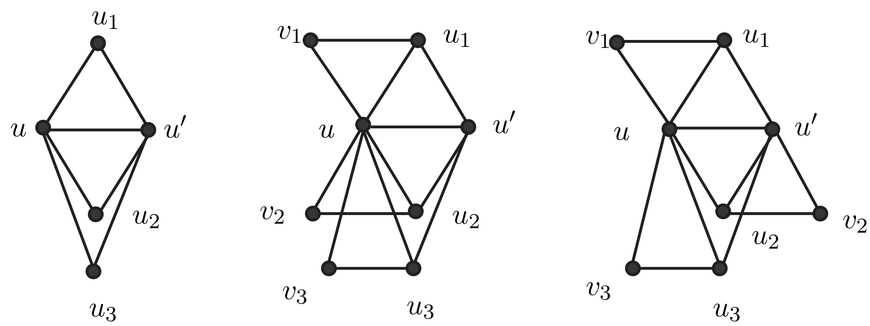
Case 1. Suppose there exists a triangle t_i, say t_3 $(1 \leq i \leq 3)$, in which the edges e_i and e'_i are not contained in any other triangles.

Let $H_2 = G - e_3$. Then there are $n-3$ triangles, $n-5$ C_4^{+}'s and $n-5-r$ 3-fans in H_2 and $TR(H_2)$ is acyclic, i.e., $TR(H_2)$ is a forest with $n-3$ vertices and 2 components. Thus H_2 is a graph of $\mathcal{F}_2^{n-3}$. Let $H_2 = T^{n_1} \cup T^{n_2}$, where $T^{n_1} \in \Gamma^{n_1}, T^{n_2} \in \Gamma^{n_2}$ and $n_1 + n_2 = n-3$.

By Proposition 1.3 *i*), $|E(\hat{H}_2)| = (2n_1+1) + (2n_2+1) = 2n-4$. Since e_3 and e'_3 are contained in t_3 only, so $e_3, e'_3 \notin E(\hat{H}_2)$. Thus $E(G) = E(\hat{H}_2) \cup \{e_3, e'_3\}$ and furthermore $V(G) = V(\hat{H}_2) \cup \{u_3\}$.

As G is 3-edge-connected, $\delta(G) \geq 3$. Thus in H_2, there are at least one edge e''_3 ($e''_3 \notin \{e_3, e'_3\}$) incident with u_3. Since each edge of G is contained in some triangle, $e''_3 \in E(\hat{H}_2)$ and thus $u_3 \in V(\hat{H}_2)$. Therefore $V(G) = V(\hat{H}_2)$. Since G is 2-connected and $\delta(G) \geq 3$, $|V(G)| = |V(\hat{H}_2)| \leq (n_1+2) + (n_2+2) - 2 = n-1$, a contradiction.

Case 2. For every triangle t_i, $i = 1,2,3$, at least one edge of e_i and e'_i is contained in another triangle. In this case, $r \geq 3$. Since $r \leq 5$, there is an edge, say e'_3, contained in no



(a) G_0 (b) G_1 (c) G_2

Figure 3: G_0, G_1 and G_2

other triangle. Let $H_3 = G - e_3'$. Then in H_3, there are $n-3$ triangles, $n-6$ C_4^+'s and when deleting edge e_3' from G, the number of 3-fans decreased by at least $r+1$, so there are at most $n-6-r$ 3-fans. Clearly $TR(H_3)$ is acyclic. Thus, $TR(H_3)$ is a forest with $n-3$ vertices, 3 components and at most $n-6-r$ 2-paths.

By Lemma 1.5, there are at least $n-9$ 2-paths in $TR(H_3)$. Thus, $n-6-r \geq n-9$, or $r \leq 3$. In this case, $r = 3$ and G contains G_1 or G_2 as a subgraph (shown in Fig. 3(b) and 3(c)) and $TR(H_3)$ has exactly $n-9$ 2-paths. In addition, when deleting e_3' from G, the number of 3-fans decreased exactly by 4. Thus, both uv_3 and u_3v_3 are contained only in one triangle uu_3v_3. So, H_3 is a graph of $\digamma_3^{n-3}$ and it can be written as $H_3 = P^{n_1} \cup P^{n_2} \cup P^{n_3}$, $P^{n_i} \in \wp^{n_i}$, $i = 1, 2, 3$, and one of P^{n_i} contains exactly one triangle, say P^{n_2}. Each edge and each vertex of G is contained in some triangle, so it is easy to see that $|E(H_3)| = |E(\hat{H}_3)|$ and $|V(H_3)| = |V(\hat{H}_3)|$. Since $|E(\hat{H}_3)| = 2n-3$, we have $E(G) = E(\hat{H}_3) \cup \{e_3'\}$ and $V(G) = V(\hat{H}_3) = V(P^{n_1}) \cup V(P^{n_2}) \cup V(P^{n_3})$.

From the structure of G_1 and G_2, we may assume that $u \in V(P^{n_1}) \cup V(P^{n_2})$. Since G is *2-connected* and $\delta(G) \geq 3$, $|V(G)| = |V(\hat{H}_3)| \leq |V(P^{n_1})| + |V(P^{n_2})| - 1 + |V(P^{n_3})| - 3 \leq n-1$, a contradiction. This completes the proof of Claim.

Since $\tau_3 = 0$, by Inequality (7), every edge except one is contained in a triangle. On the other hand, by Lemma 2.3, $TR(G)$ is a tree and contains exactly $n-4$ 2-path. Therefore, by Lemma 1.5, $TR(G)$ is an $(n-2)$-path.

Since G is 2-connected, $|V(G)| = n$ and $|E(G)| = 2n-2$, every vertex is contained in triangle, i.e. $V(\hat{G}) = V(G)$. Thus, G is a graph satisfying the three conditions of Lemma 1.4, so every 3-element bond of G is trivial. Therefore, there are $n-3$ 3-degree vertices in G.

Let F_k be a maximum fan in G, if $k = n-2$, after adding the strap edge to G, the number of triangles of G will increase by 1, it is a contradiction since G has exactly $n-2$ triangles. Therefore, $3 \leq k \leq n-3$. We use the same labeling as in Fig. 2. Since $3 \leq k \leq n-3$ and

F_k is maximum, there exists a triangle t_1 using a brim non-spoke-edge of F_k, without loss of generality, we assume that $v_{k-1}v_k$ is contained in t_1 and $t_1 = v_{k-1}v_kw_1$. By adding t_1 to F_k, the degrees of vertices v_{k-1} and v_k increase to 4 and 3, respectively.

Obviously, w_1 is not the 2-degree vertex of $\hat{G}$, since otherwise, the number of C_4's without chord is more than 2. So there exists a triangle t_2 containing w_1. If $v_{k-1}w_1 \in t_2$, since there are exactly 3 vertices of G with degree more than 3 and $\hat{G} = P_{max}^{n-2}$, then the 2-degree vertices of $\hat{G}$ can not be adjacent to v_{k-1}. Thus, in $\hat{G}$, the distance between the two 2-degree vertices is more than 3. Since $G \in \wp^{n-2}$ and $\hat{G} = P_{max}^{n-2}$, every C_4 has one chord in $\hat{G}$. Thus the edge joining the two 2-degree vertices must be in a C_4 containing no chord. Therefore, the distance of the two 2-degree vertices in $\hat{G}$ is exactly 3, this is a contradiction. Therefore $v_kw_1 \in t_2$. It is simple to see that G is isomorphic to $W_h(k_1', k_2')$, $k_1' = k$ and $k_2' = n-2-k$.

Finally, we show that $k_1' = k_1$ and $k_2' = k_2$. Otherwise, assume $k_2 < k_2'$, i.e., $k_2 < k_2' \le k_1' < k_1$, we consider the coefficient of $x^{k_2+2}y^{2k_2+3}$. By the remark after Proposition 2.2, $[x^{k_2+2}y^{2k_2+3}]F(W_h(k_1,k_2);x,y) \ge n-k_2-1$. However, since $k_2+2 \le k_2'+1$, $[x^{k_2+2}y^{2k_2+3}]F(W_h(k_1',k_2');x,y) = n-k_2-2$, a contradiction. Therefore, $k_1' = k_1$ and thus $k_2' = k_2$. This completes the proof of the theorem. □

Remark. The twisted wheels studied in [4] and the double half-wheels defined in this paper have a very similar structure. However, the T-uniqueness of one class can not be deduced from the T-uniqueness of another even though the proof ideas and techniques overlap quite a bit. It will be interesting to characterize a large family of graphs consisting of mostly triangles whose T-uniqueness can be dealt by triangle-induced subgraphs uniformly.

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