The Interplanetary Superhighway

Chaotic transport through the solar system

Richard Taylor

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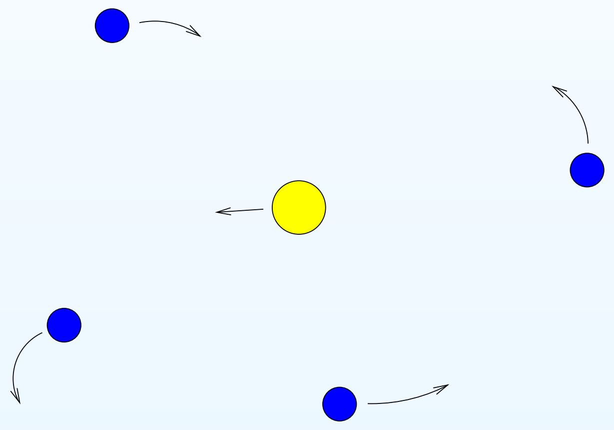
TRU

The N-Body Problem

N masses interact via mutual gravitational attraction.



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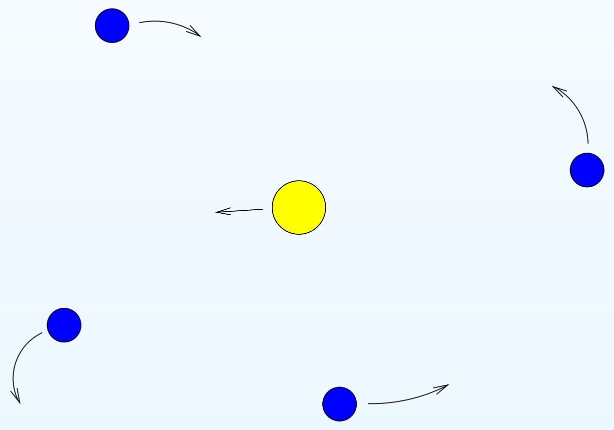

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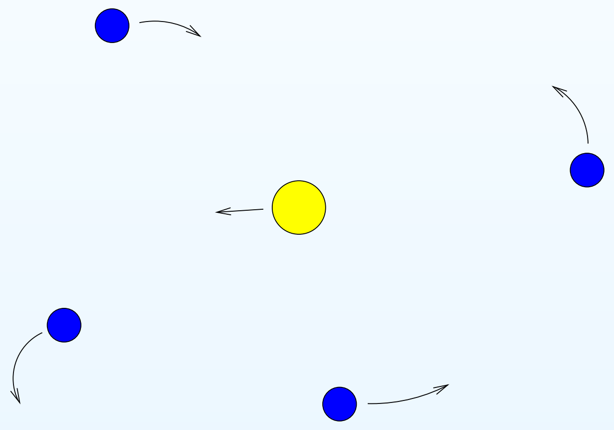

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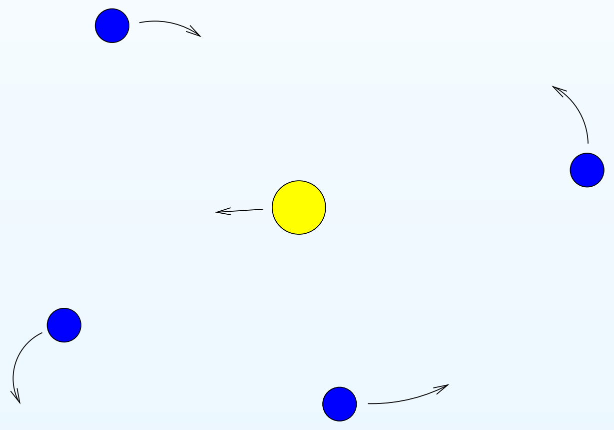
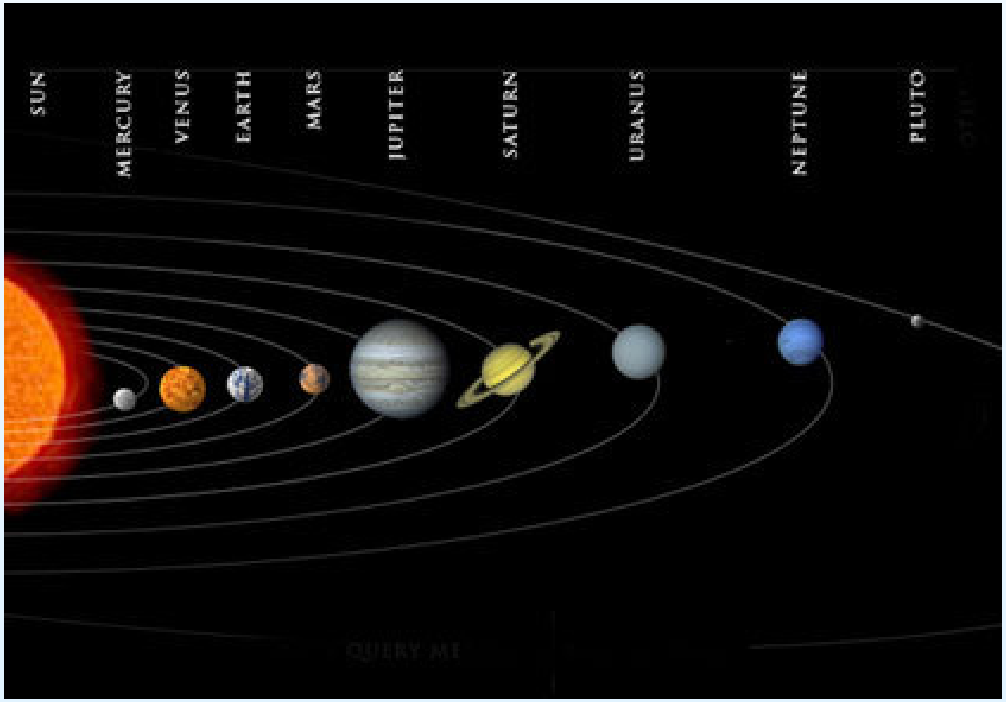


- predict the evolution for all time?
- characterize long-term behavior?
- determine stability?

Some History...

- Kepler ($\sim$ 1600) – empirical “laws” of planetary motion



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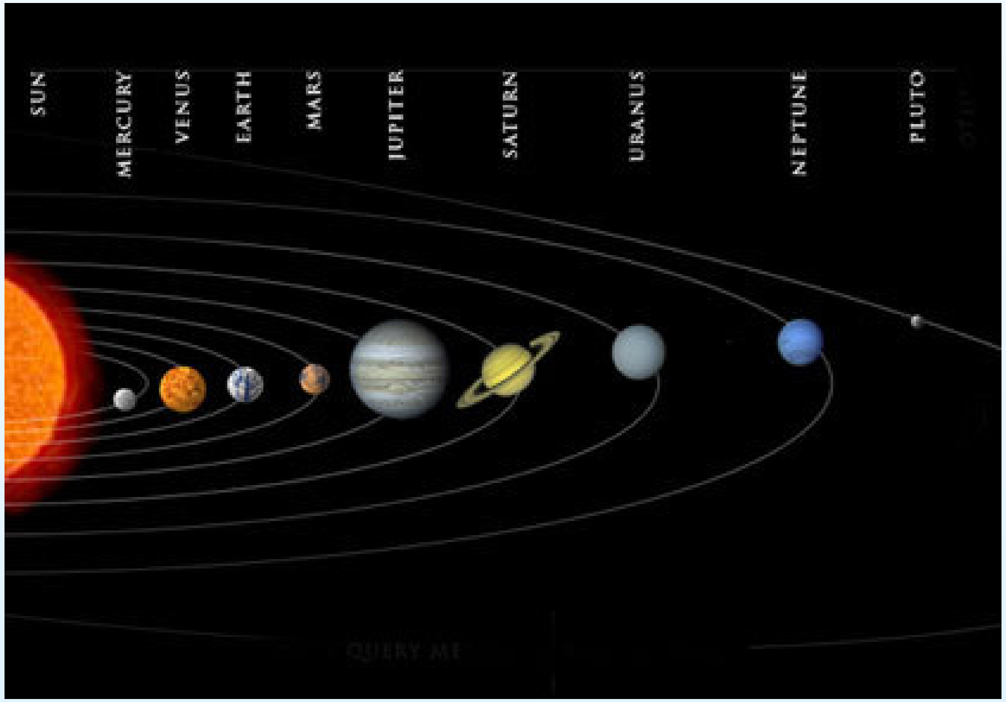

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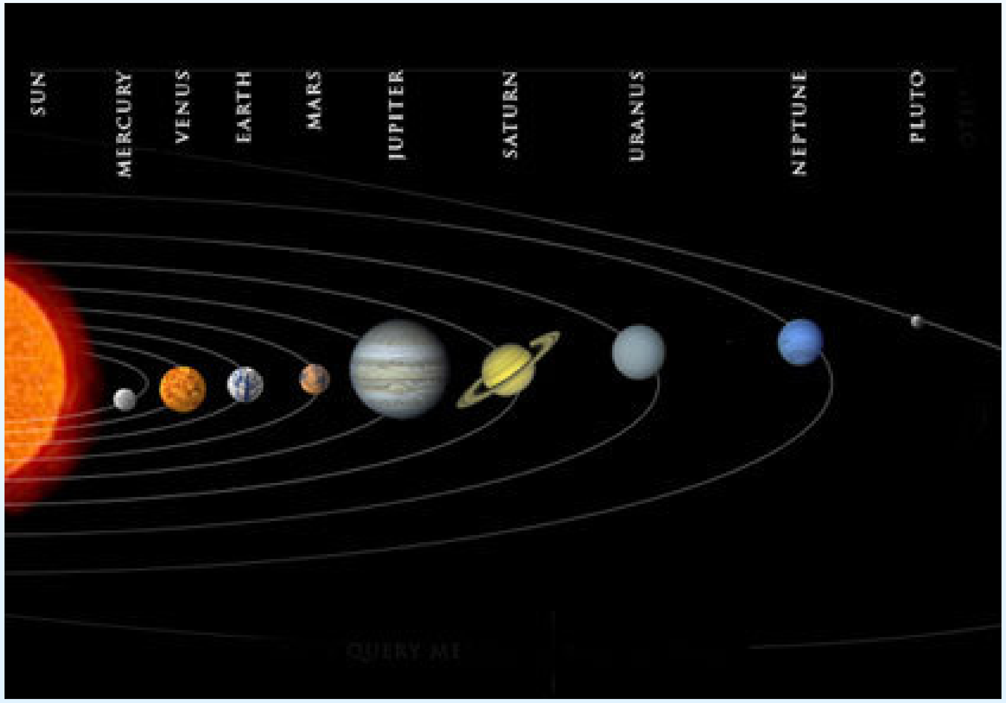

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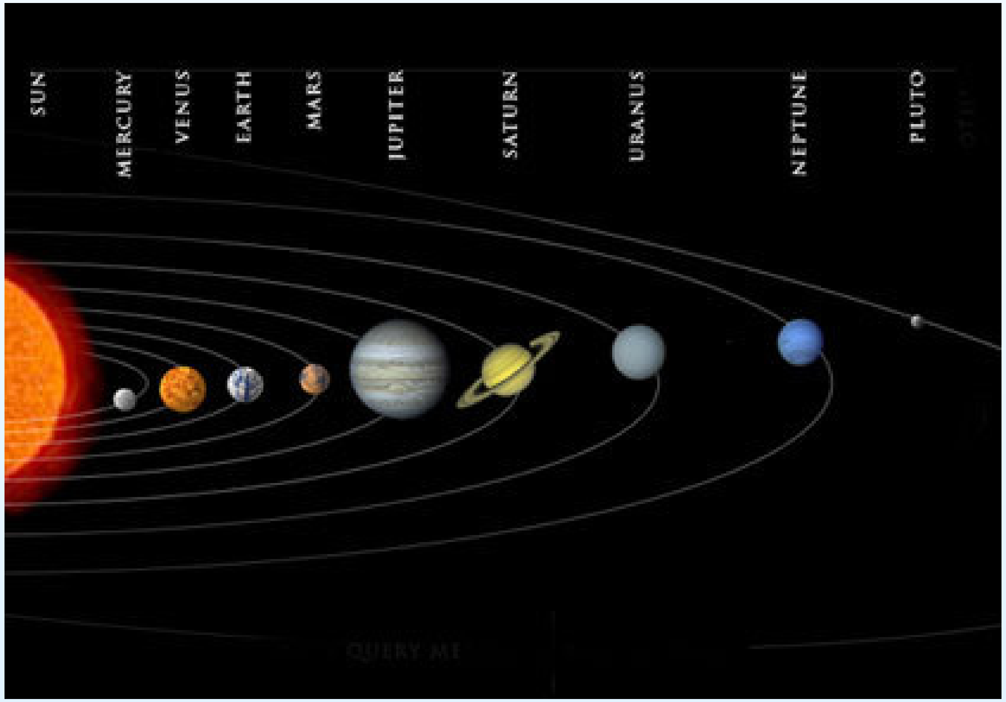
- ca. 2000 – basic problems (e.g. stability) are still open



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- Expensive: $1 million to take 1 lb to the moon
- Doesn't take advantage of N-body dynamics

N-Body Problem: Equations of Motion

$$\mathbf{r}_i'' = \sum_{1 \le j \le N, j \ne i} G m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

where $\mathbf{r}_i \in \mathbb{R}^3 =$ position of mass m_i ($i = 1, \ldots, N$).

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Phase space is $6N$-dimensional: 3 position + 3 velocity coordinates for each of N objects.

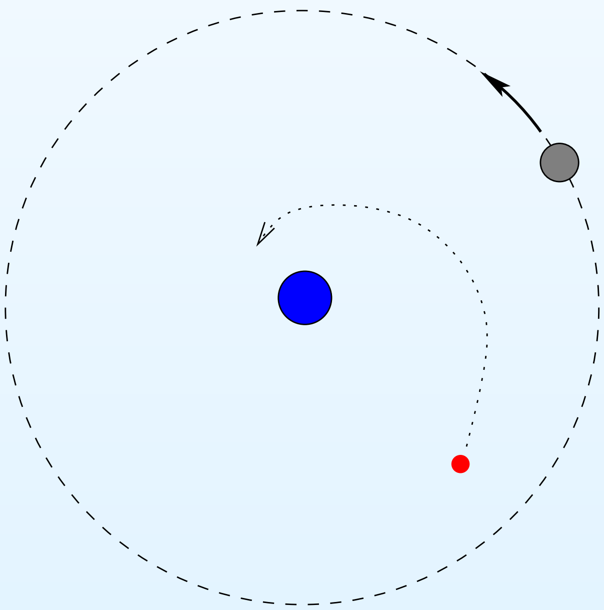
Lots of dimensions. Lots of parameters. hrrmmm....

Restricted 3-Body Problem

Planar Circular Restricted 3-Body Problem = PCR3BP

Simplifying assumptions:

- all objects move in a single plane



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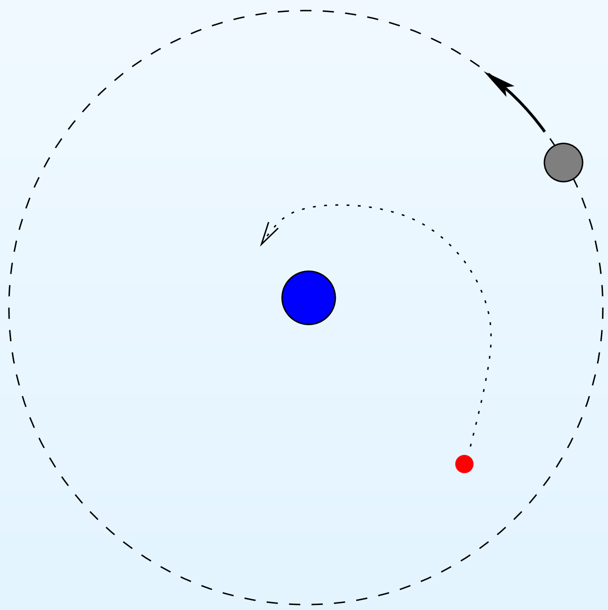

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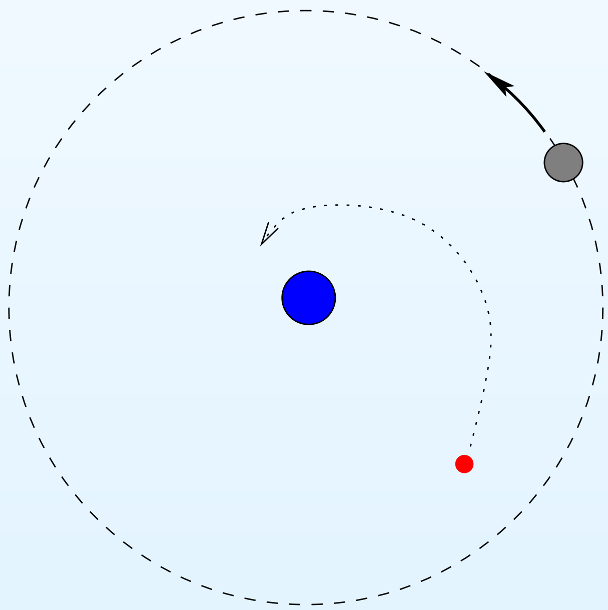

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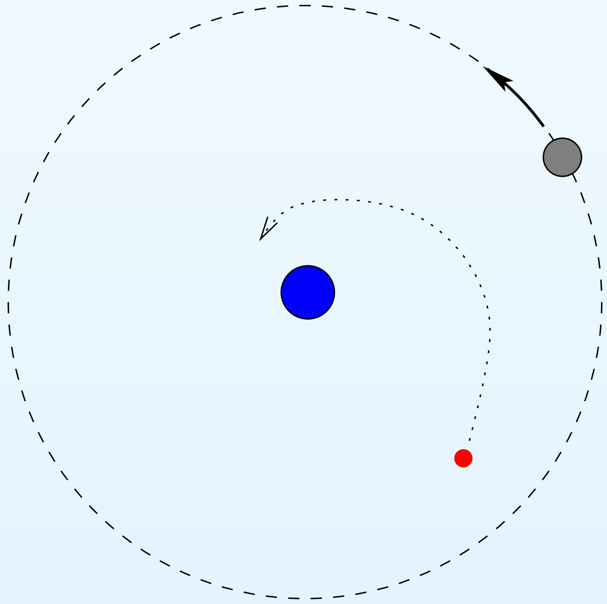

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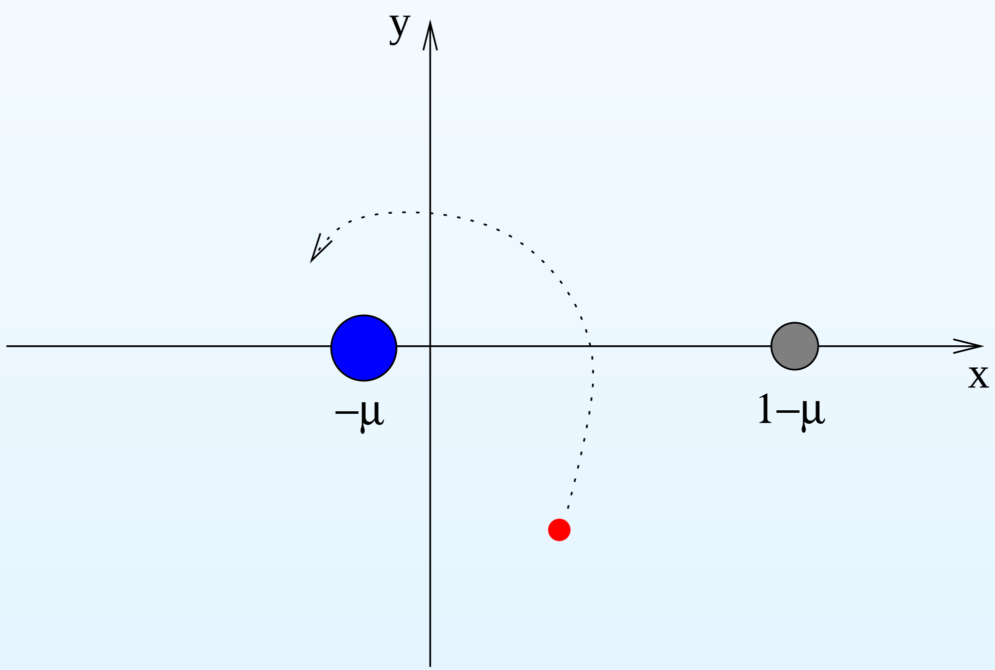
PCR3BP: Rotating Coordinate System

- coord. system rotates with circular orbit ($\omega = 1$)



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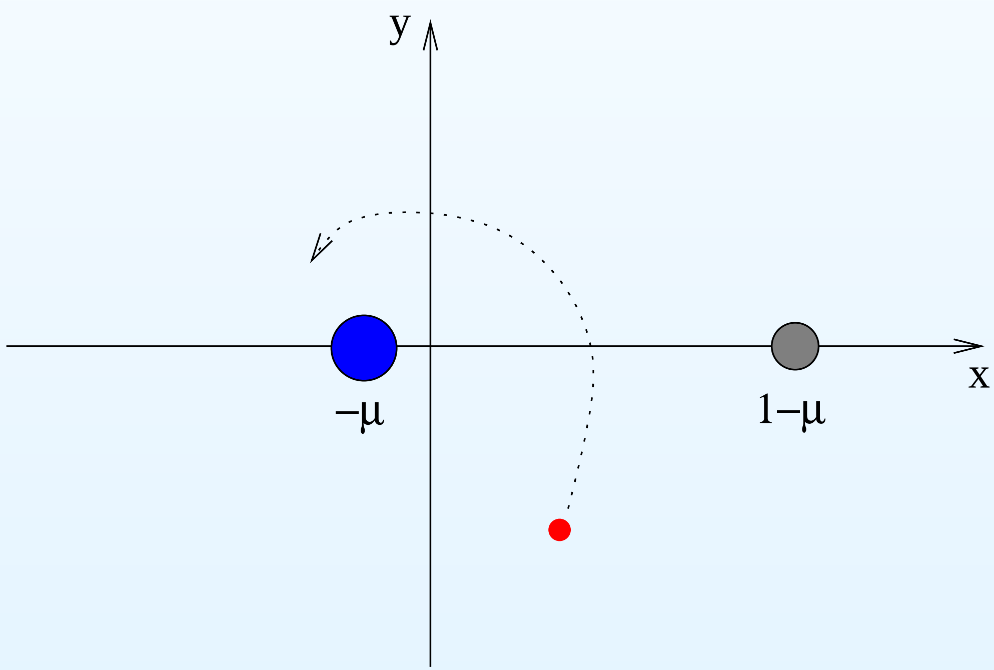

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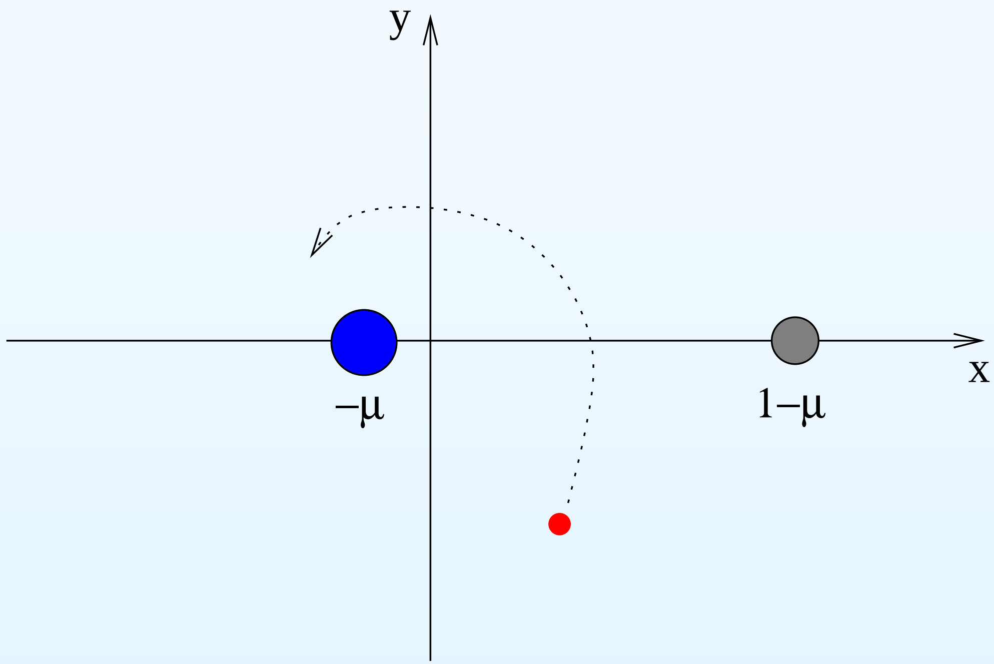

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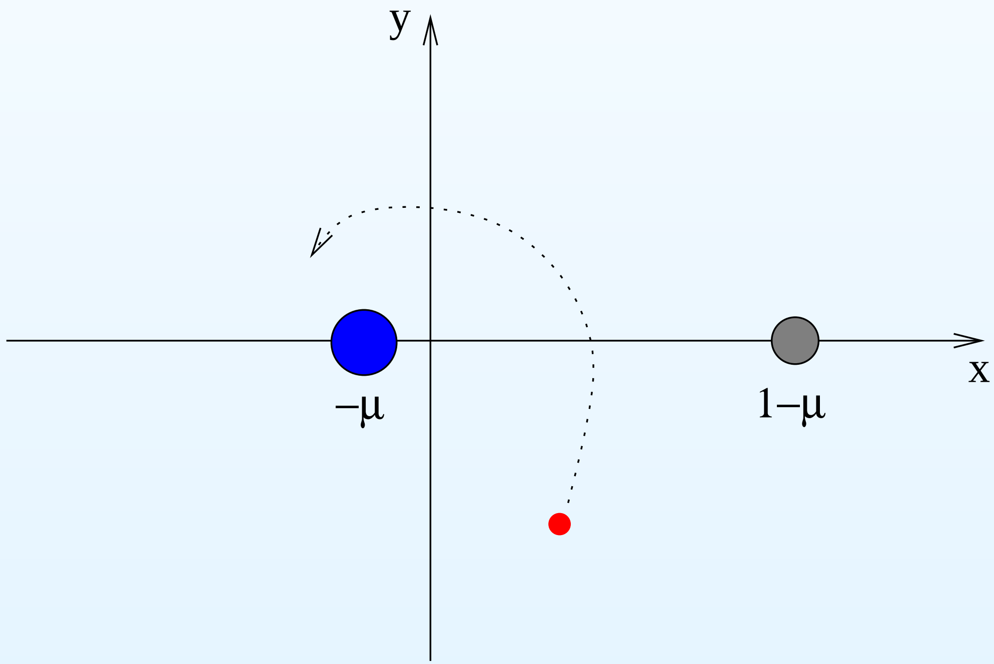

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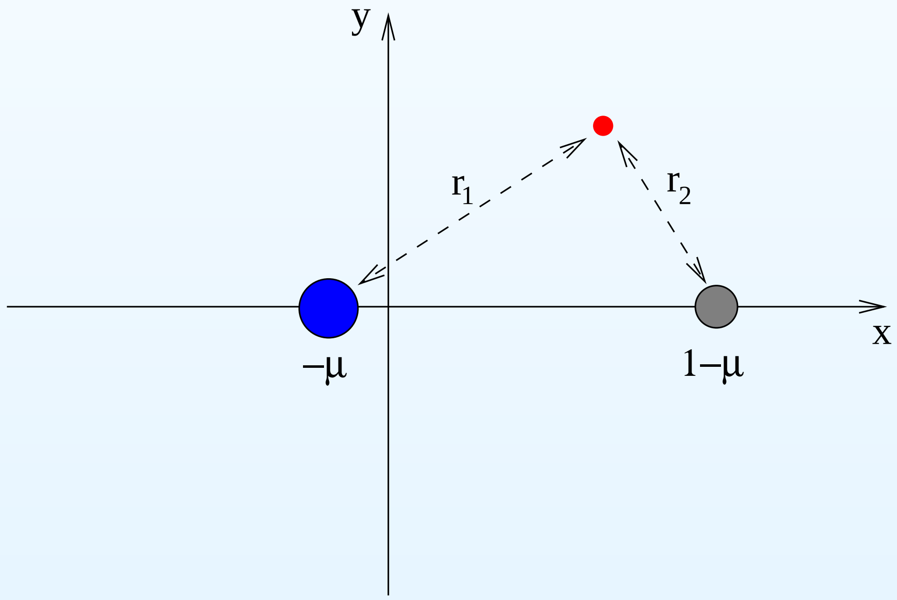

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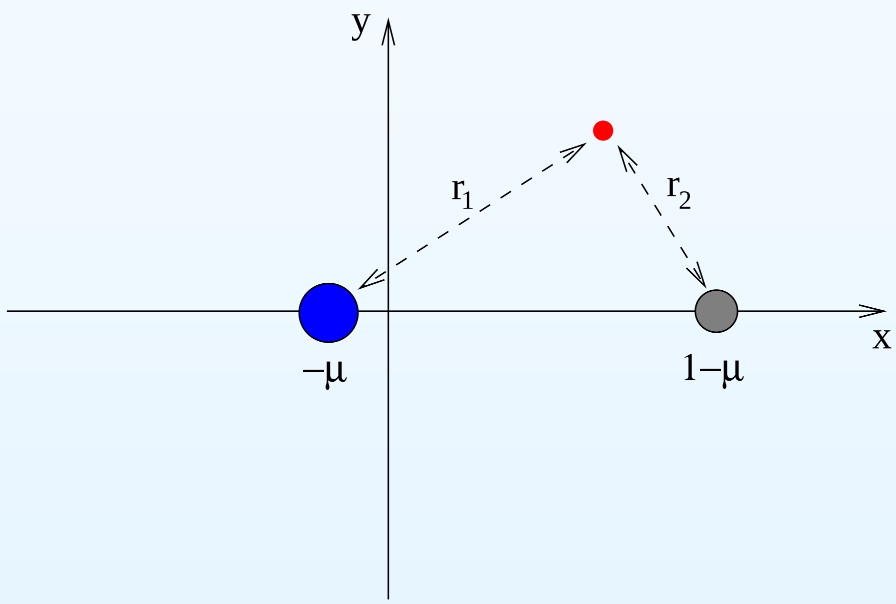

$$\Longrightarrow \;\; \mathbf{x}' = f(\mathbf{x}), \quad \mathbf{x} = (x, y, x', y') \in \mathbb{R}^4$$

Visualizing Solutions of the 3-Body Problem

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Any solution specifies a curve $\{\mathbf{x}(t) : t \geq 0\} \subset \mathbb{R}^4$. Hard to visualize.

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But... energy $J(\mathbf{x})$ is conserved $\implies$ a given solution is restricted to a particular 3-manifold of constant energy $J(\mathbf{x}) = C$. Restricting our attention to orbits on this manifold, the problem is reduced to 3 dimensions.

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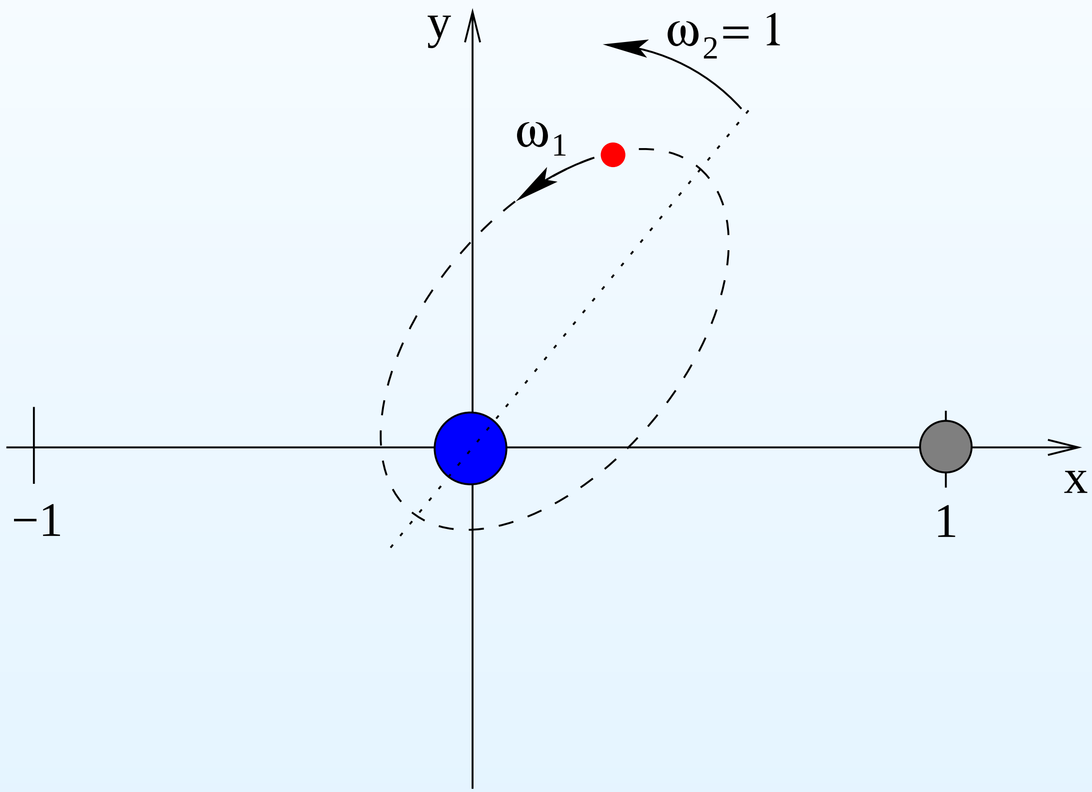
Special Case $\mu = 0$

Back to the 2-body problem. Kepler orbit gives a precessing ellipse with two rotational frequencies.



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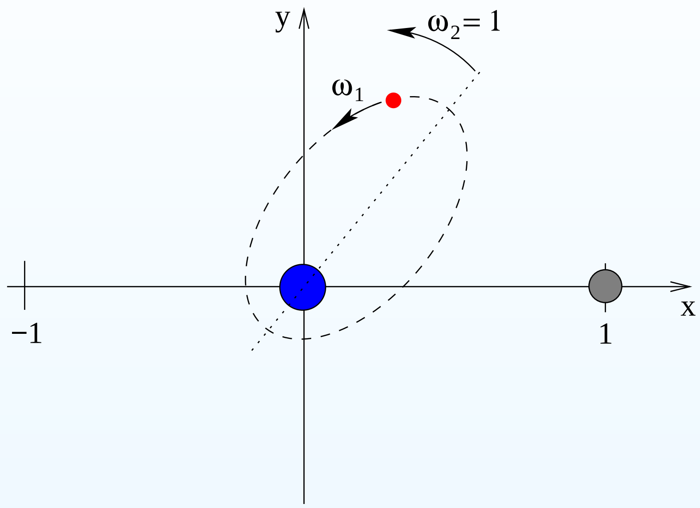
Invariant Tori

In physical space the orbit is a precessing ellipse. The orbit is completely parametrized by two angles.



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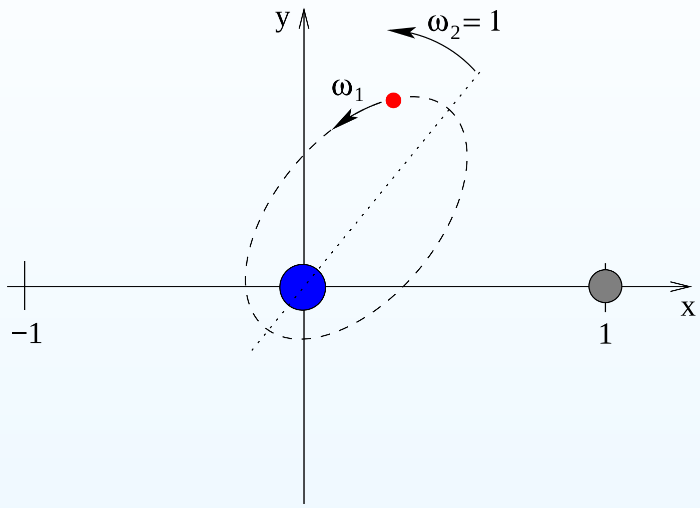

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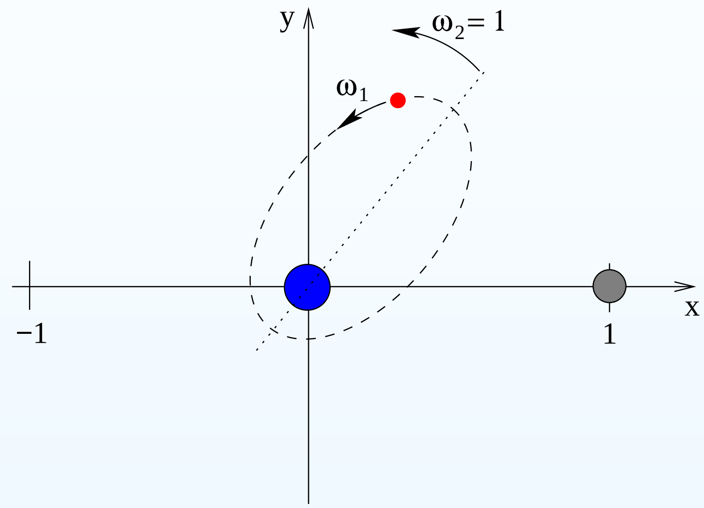

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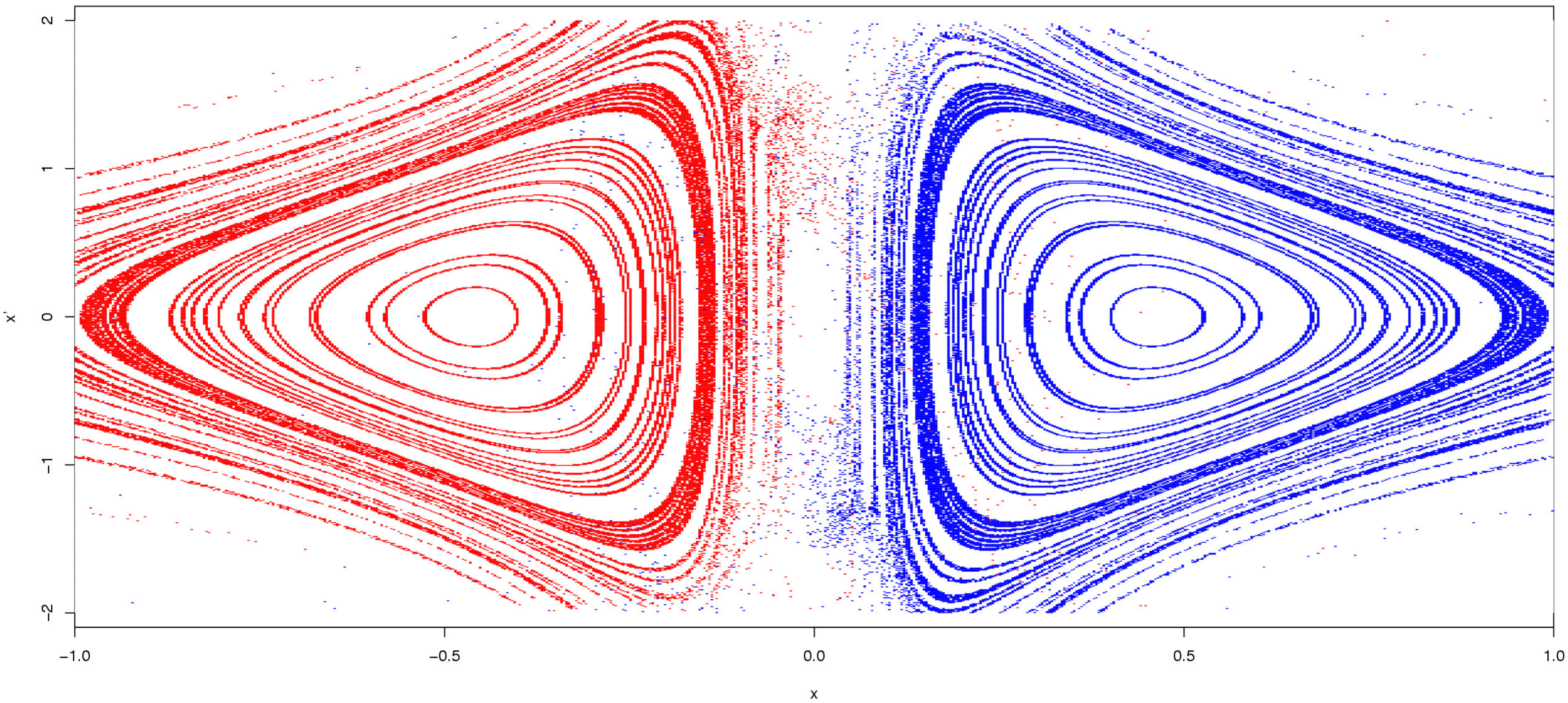
Different initial conditions give a curve $\mathbf{x}(t)$ on a different torus. In this way we get a family of nested invariant tori that foliate the 3-manifold of constant energy.

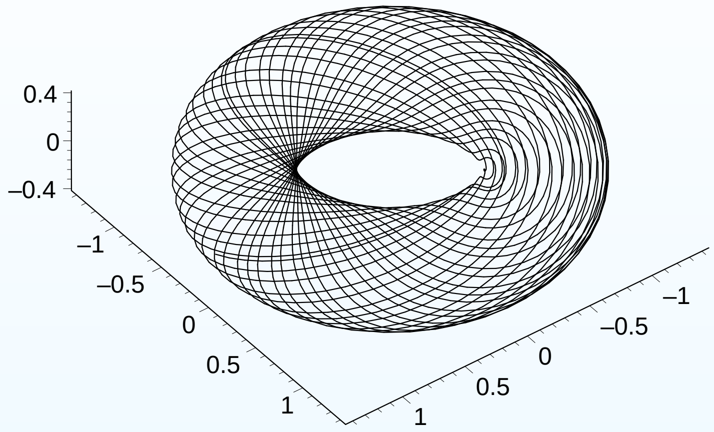
<plot>
<simulation>

Poincaré Section ($\mu = 0$)



Resonance

If $\frac{\omega_1}{\omega_2}$ is irrational then the torus is covered with a dense orbit (*quasi-periodicity*).



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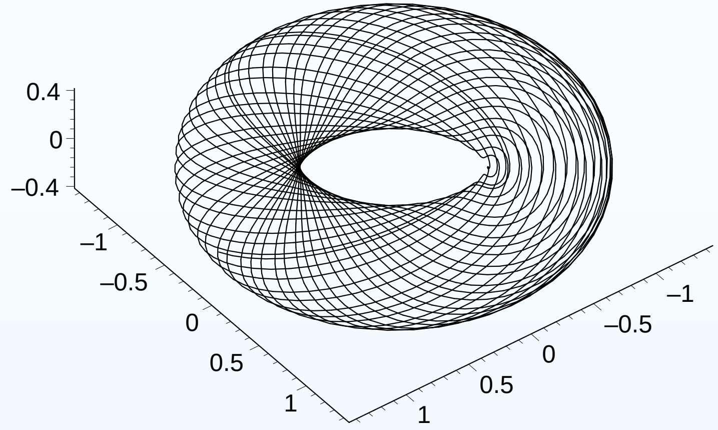
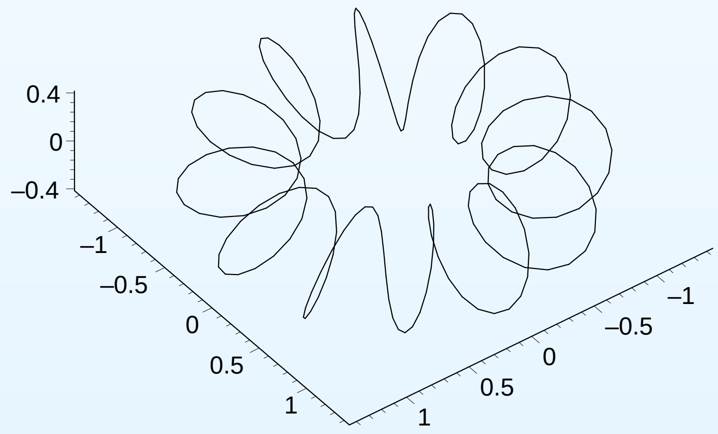


If $\frac{\omega_1}{\omega_2} = \frac{m}{n}$ is a rational number the orbit is periodic: the spacecraft completes m revolutions just as the planet completes n revolutions (*resonance*).



Perturbed Case $0 < \mu \ll 1$

Theorem (Kolmogorov-Arnold-Moser). *For all sufficiently small* μ *the perturbed system* $\mathbf{x}' = f(\mathbf{x})$ *has a set of invariant tori, each of which is covered with a dense orbit. This set has positive Lebesgue measure.*

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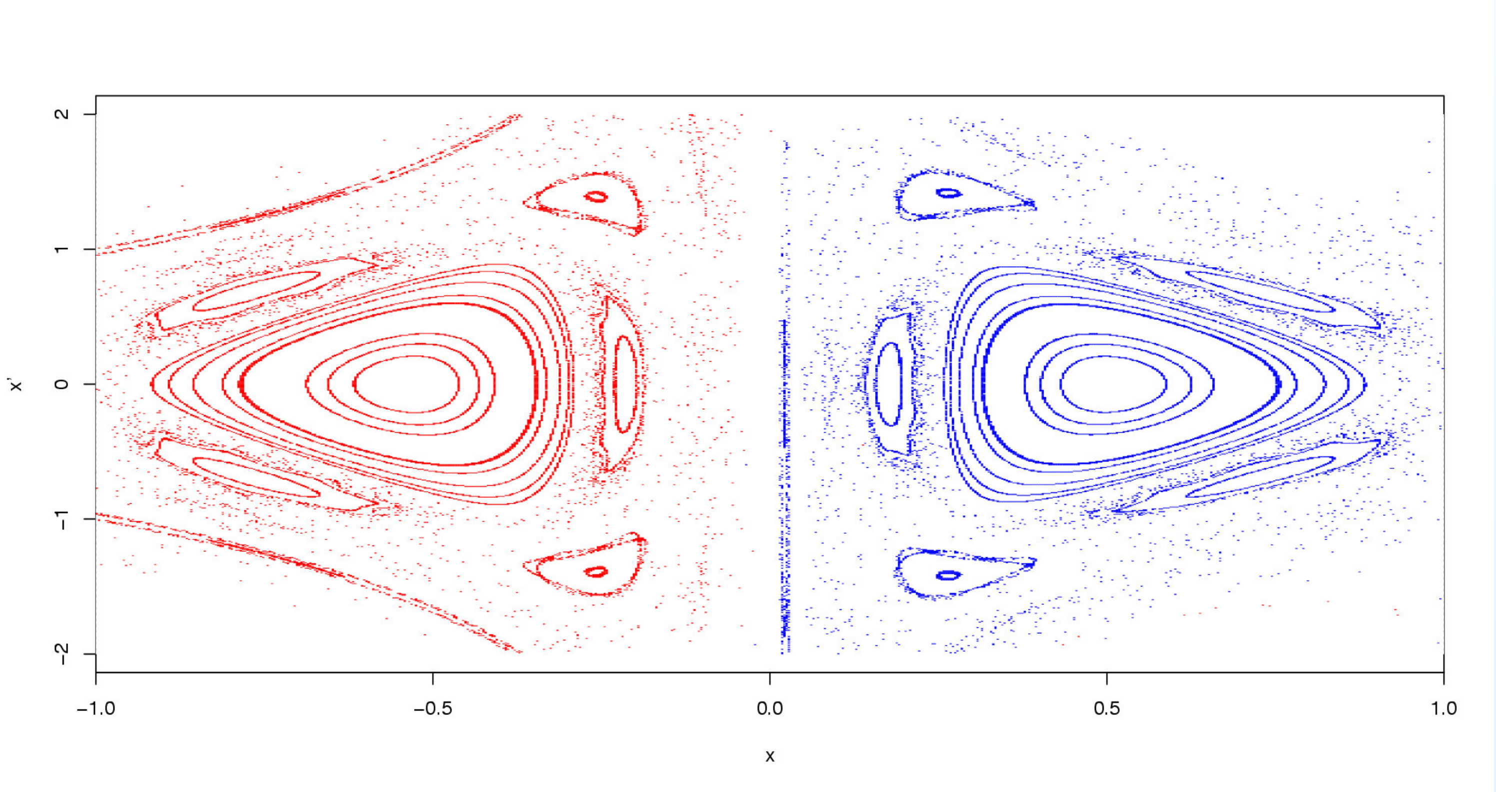
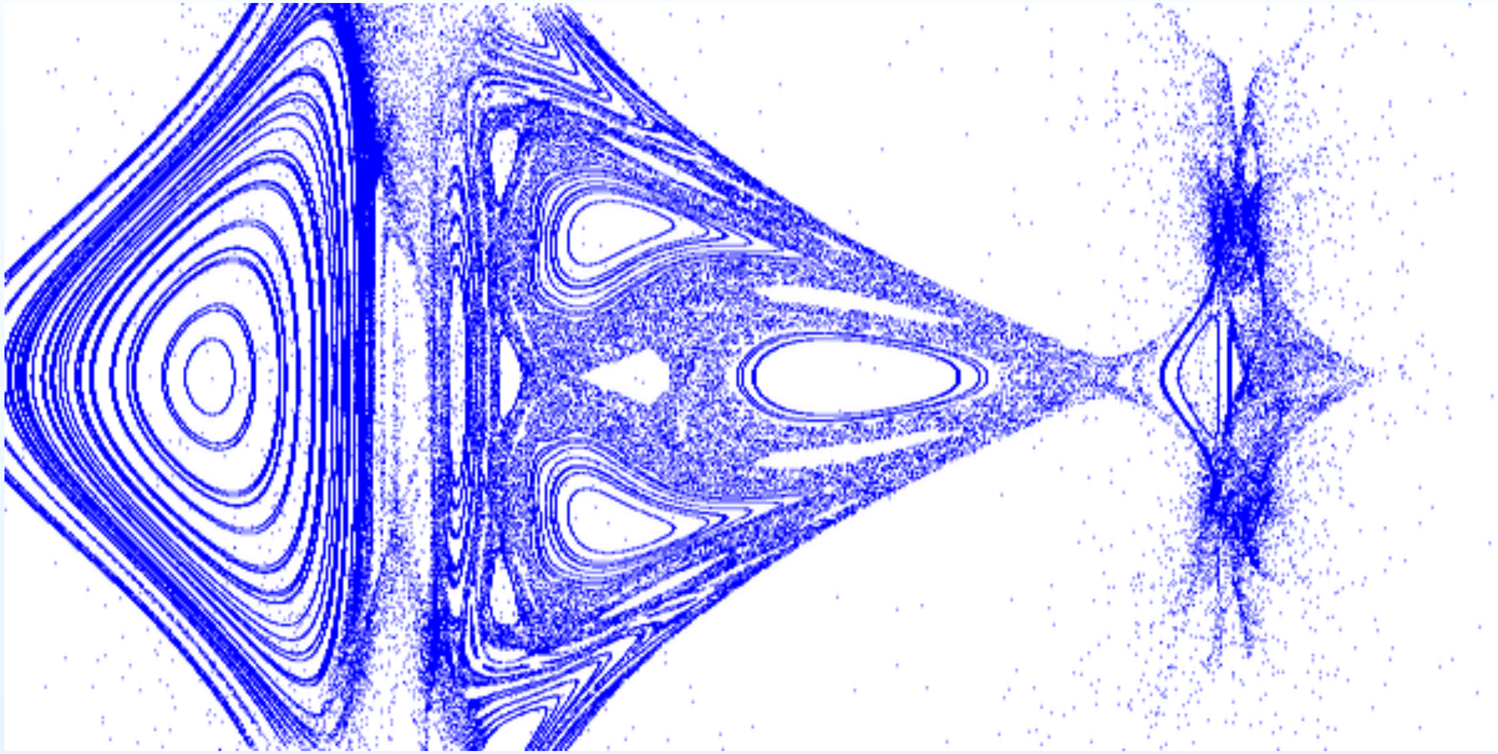


Orbits Near Resonance

We know orbits far from resonance are stuck on invariant tori.
What about the orbits near resonance?



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Homoclinic Chaos

Theorem (Smale-Birkhoff). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that p is a hyperbolic fixed point and there exists a point $q \neq p$ of transversal intersection between the stable and unstable manifolds of p. Then there is a hyperbolic invariant set $\Lambda \subset \mathbb{R}^n$ on which f is topologically equivalent to a subshift of finite type.*

$\Lambda \subset \mathbb{R}^4$ is a set on and near which dynamics are “chaotic". Some consequences...

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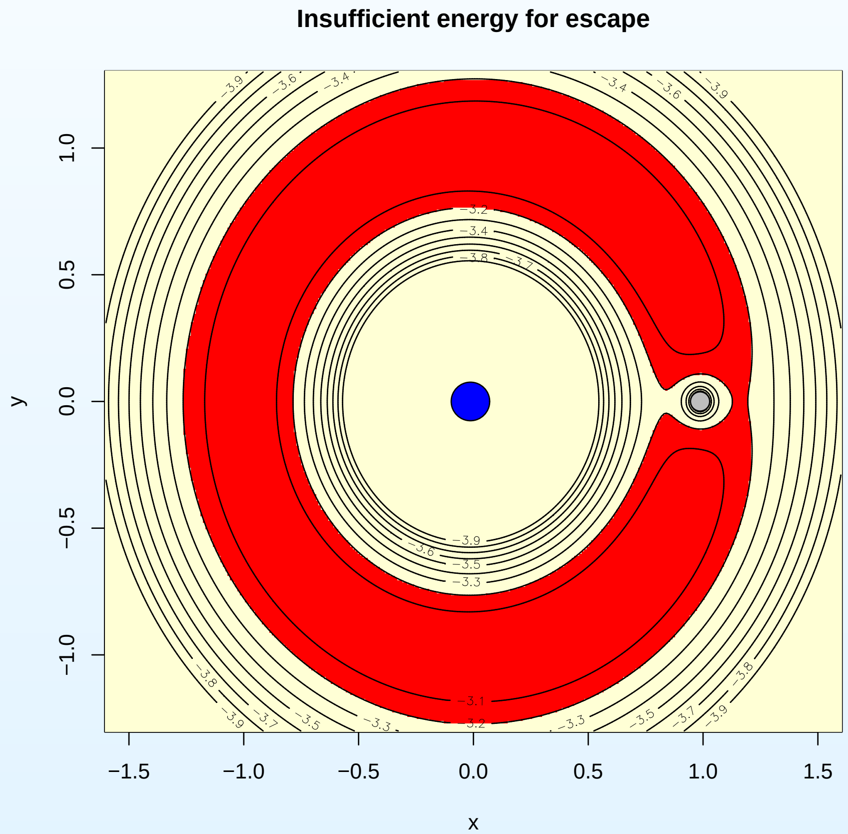
- The dynamics on Λ have sensitive dependence on initial conditions
- Λ is a Cantor set (uncountable, measure-zero) — a fractal

Low-Energy Escape from Earth

Contour Plots of Potential Energy $\Omega(x,y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

 = "forbidden region"



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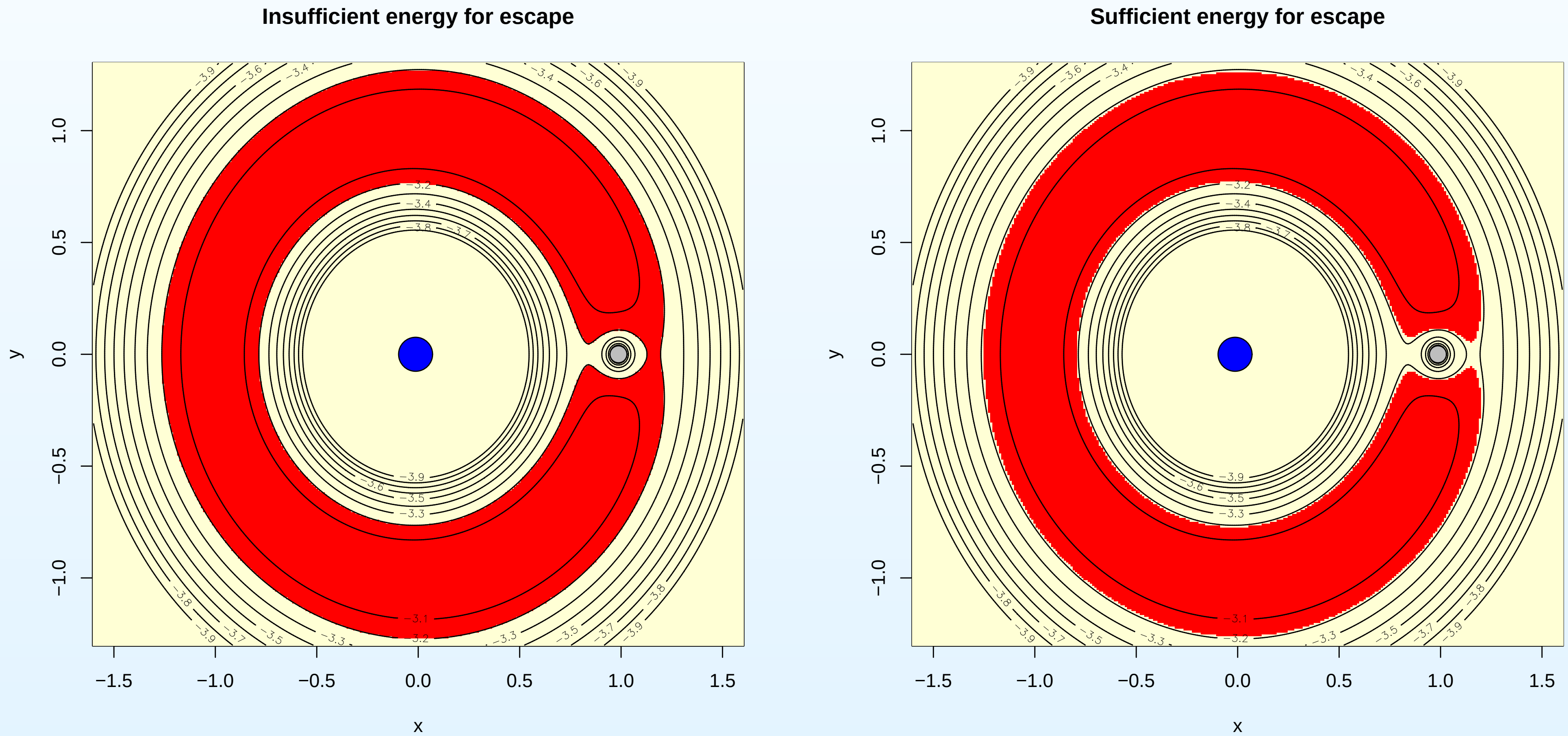
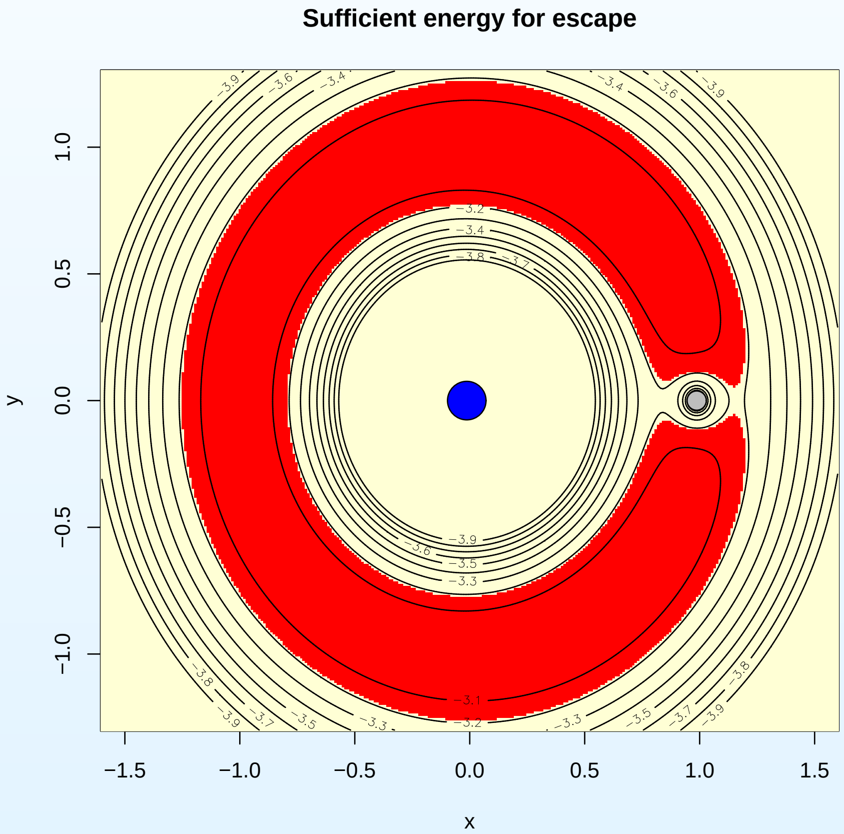


Chaos and Low-Energy Escape from Earth

So... to escape Earth with near minimal energy, you must pass through the “bottleneck" region, and hence exit on a chaotic trajectory.



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Other applications

- Predicting chemical reaction rates (3-body problem models an electron shared between atoms)