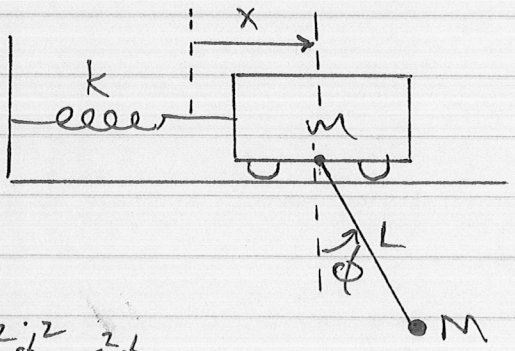
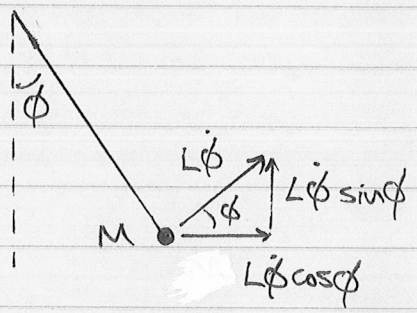
7.31



a) $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left[\left(\dot{x} + L\dot{\phi}\cos\phi\right)^2 + \left(L\dot{\phi}\sin\phi\right)^2\right]$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left[\dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2\cos^2\phi + L^2\dot{\phi}^2\sin^2\phi\right]$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left[\dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2\right]$



$U = \frac{1}{2}kx^2 - MgL\cos\phi$

$\rightarrow L = T - U$

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M\left[\dot{x}^2 + 2L\dot{x}\dot{\phi}\cos\phi + L^2\dot{\phi}^2\right] - \frac{1}{2}kx^2 + MgL\cos\phi$

Equations of motion:

$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \rightarrow -kx = \frac{d}{dt}\left[(m+M)\dot{x} + ML\dot{\phi}\cos\phi\right]$

$\rightarrow -kx = (m+M)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi \quad (1)$

$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \rightarrow -ML\dot{x}\dot{\phi}\sin\phi - MgL\sin\phi = \frac{d}{dt}\left[ML\dot{x}\cos\phi + ML^2\dot{\phi}\right]$

$\rightarrow -ML\left(\dot{x}\dot{\phi} + g\right)\sin\phi = ML\ddot{x}\cos\phi - ML\dot{x}\dot{\phi}\sin\phi + ML^2\ddot{\phi} \quad (2)$

$(1) \rightarrow (m+M)\ddot{x} + ML\ddot{\phi}\cos\phi = ML\dot{\phi}^2\sin\phi - kx$

$(2) \rightarrow \ddot{x}\cos\phi + L\ddot{\phi} = \cancel{\dot{x}\dot{\phi}\sin\phi} - \cancel{\dot{x}\dot{\phi}\sin\phi} - g\sin\phi$

So the eqs. of motion are:

$$\boxed{\begin{aligned}(m+M)\ddot{x} + L\ddot{\phi}\cos\phi &= L\dot{\phi}^2\sin\phi - kx \\ \ddot{x}\cos\phi + L\ddot{\phi} &= -g\sin\phi\end{aligned}}$$

b) if both x and ϕ are small:

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \cdots \approx \phi$$

$$\cos\phi = 1 - \frac{\phi^2}{2} + \cdots \approx 1$$

$$\begin{cases} (m+M)\ddot{x} + L\ddot{\phi} = \underbrace{L\phi\dot{\phi}^2}_{\text{negligible for small } \phi,\,\dot{\phi}} - kx \\ \ddot{x} + L\ddot{\phi} = -g\phi \end{cases}$$

$$\rightarrow \boxed{\begin{aligned}(m+M)\ddot{x} + L\ddot{\phi} &= -kx \\ \ddot{x} + L\ddot{\phi} &= -g\phi\end{aligned}}$$