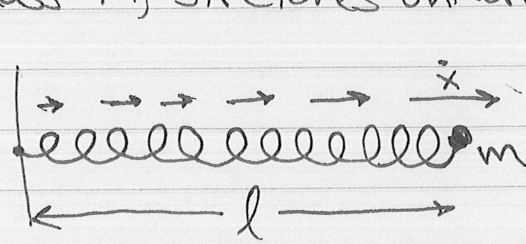
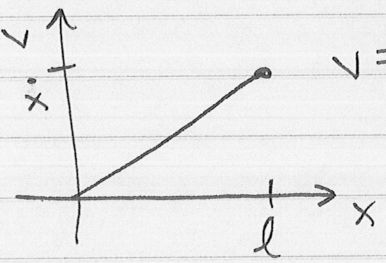
<u>13.6</u> Spring, mass M, stretches uniformly:

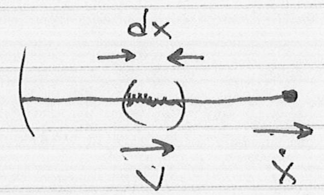


Spring velocity will be a linear function of x:



$$v = \left(\frac{x}{\ell}\right)\dot{x}$$

Segment of length dx at x has K.E.:



$$\begin{aligned} dT &= \frac{1}{2}(dm)\,v^2 \\ &= \frac{1}{2}\left(\frac{M}{\ell}dx\right)\left(\frac{x}{\ell}\dot{x}\right)^2 \\ &= \frac{1}{2}\frac{M\dot{x}^2}{\ell^3}x^2\,dx \end{aligned}$$

so the spring's total K.E. is

$$T = \int dT = \int_0^\ell \frac{1}{2}\frac{M\dot{x}^2}{\ell^3}x^2\,dx = \frac{1}{2}\frac{M\dot{x}^2}{\ell^3}\left(\frac{1}{3}x^3\right)\Bigg|_{x=0}^{x=\ell}$$

$$= \frac{1}{2}\left(\frac{M}{3}\right)\dot{x}^2$$

Thus the mass-spring system has:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{M}{3}\right)\dot{x}^2 \longrightarrow L = T - U$$

$$U = \frac{1}{2}kx^2 \qquad\qquad = \frac{1}{2}\left(m + \frac{M}{3}\right)\dot{x}^2 - \frac{1}{2}kx^2$$

momentum:

$$p = \frac{\partial L}{\partial \dot{x}} = \left(m + \frac{M}{3}\right)\dot{x} \longleftrightarrow \dot{x} = \frac{p}{m + M/3}$$

Hamiltonian:

$$H = p\dot{x} - L$$

$$= p\dot{x} - \overbrace{\frac{1}{2}\left(m + \frac{M}{3}\right)\dot{x}^2}^{p\dot{x}} + \frac{1}{2}kx^2$$

$$= \frac{1}{2}p\dot{x} + \frac{1}{2}kx^2$$

$$= \frac{\frac{1}{2}p^2}{m + M/3} + \frac{1}{2}kx^2$$

Equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m + M/3} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

$$\rightarrow \ddot{x} = \frac{\dot{p}}{m + M/3} = -\frac{k}{m + M/3}\,x$$

Solving the DE gives:

$$x(t) = e^{rt} \rightarrow r^2 + \frac{k}{m + \frac{M}{3}} = 0 \rightarrow r = \pm i \underbrace{\sqrt{\frac{k}{m + M/3}}}_{\omega}$$

So $x(t) = A\cos\omega t + B\sin\omega t$