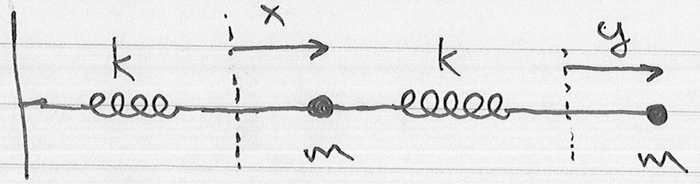
11.5



x, y are relative to rest positions of masses.

$$\left.\begin{aligned} T &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 \\ U &= \tfrac{1}{2}kx^2 + \tfrac{1}{2}k(y-x)^2 \end{aligned}\right\} \Rightarrow \begin{aligned} L &= T - U \\ &= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}kx^2 \\ &\quad - \tfrac{1}{2}k(y-x)^2 \end{aligned}$$

Equations of motion:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Rightarrow m\ddot{x} = -kx + k(y-x) \qquad (1)$$

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Rightarrow m\ddot{y} = -k(y-x) \qquad (2)$$

are already linear. In matrix form:

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}}_{\ddot{\underline{x}}} = \underbrace{\begin{bmatrix} -2k & k \\ k & -k \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\underline{x}}$$

Assume $\underline{x}(t) = \underline{z}\, e^{i\omega t}$:

$$\Rightarrow (K + \omega^2 M)\underline{z} = \underline{0} \quad \text{with } K + \omega^2 M = \begin{bmatrix} \omega^2 m - 2k & k \\ k & \omega^2 m - k \end{bmatrix}$$

Non-trivial solutions require:

$$\begin{aligned} 0 = \det(K + \omega^2 M) &= (\omega^2 m - 2k)(\omega^2 m - k) - k^2 \\ &= m^2\omega^4 - 3km\omega^2 + k^2 \end{aligned}$$

$$\Rightarrow \omega^2 = \frac{3km \pm \sqrt{9k^2m^2 - 4k^2m^2}}{2m^2} = \frac{k}{m}\;\frac{3\pm\sqrt{5}}{2}$$

$$\Rightarrow \boxed{\omega = \omega_0 \cdot \sqrt{\frac{3\pm\sqrt{5}}{2}}} \text{ where } \omega_0 = \sqrt{\frac{k}{m}}.$$

Now solve for the modes... solve $(K+\omega^2 M)\underline{z}=\underline{0}$ ...

1) case $\omega^2 = \frac{k}{m}\cdot\frac{3+\sqrt{5}}{2}$:

$$\underline{0} = \left(K + \frac{k}{m}\cdot\frac{3+\sqrt{5}}{2}M\right)\underline{z} = \begin{bmatrix} \frac{\sqrt{5}-1}{2}\cdot\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{\sqrt{5}+1}{2}\cdot\frac{k}{m} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\xrightarrow{RREF} \left(\begin{array}{cc|c} \frac{\sqrt{5}-1}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{\underline{z} = \begin{bmatrix} -2 \\ \sqrt{5}-1 \end{bmatrix}}$$

Motion is out-of-phase, amplitude of x is larger.

2) case $\omega^2 = \frac{k}{m}\cdot\frac{3-\sqrt{5}}{2}$:

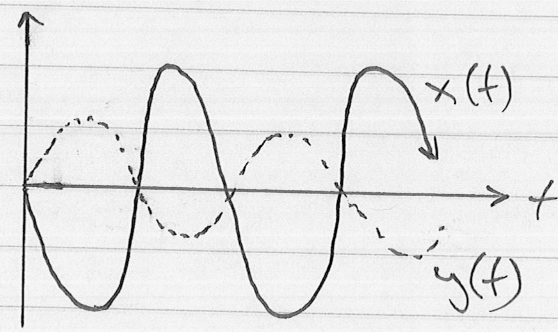
$$\underline{0} = \left(K + \frac{k}{m}\cdot\frac{3-\sqrt{5}}{2}M\right)\underline{z} = \begin{bmatrix} \frac{-1-\sqrt{5}}{2}\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{1-\sqrt{5}}{2}\frac{k}{m} \end{bmatrix}\underline{z}$$

$$\xrightarrow{RREF} \left(\begin{array}{cc|c} \frac{1+\sqrt{5}}{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{\underline{z} = \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix}}.$$

Motion is in-phase, lower frequency than other mode; amplitude of y is larger than x.

Summary:

mode 1, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ \sqrt{5}-1 \end{bmatrix} e^{i\omega_1 t}$, $\omega_1 = \omega_0 \sqrt{\frac{3+\sqrt{5}}{2}}$



mode 2, $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1+\sqrt{5} \end{bmatrix} e^{i\omega_2 t}$, $\omega_2 = \omega_0 \sqrt{\frac{3-\sqrt{5}}{2}}$