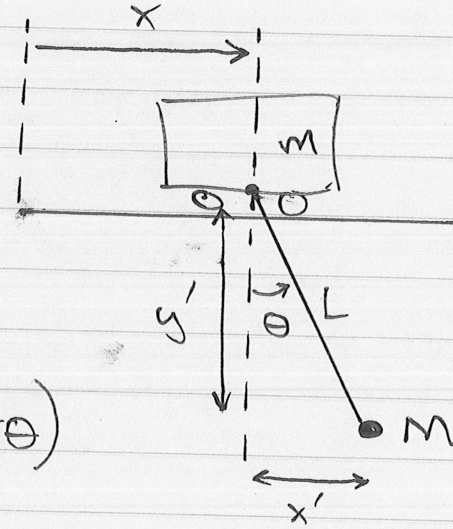
11.28



$$T = \frac{1}{2}M\left(\left[\dot{x} + L\dot{\theta}\cos\theta\right]^2 + \left[-L\dot{\theta}\sin\theta\right]^2\right) + \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}M\left(\dot{x}^2 + 2\dot{x}\dot{\theta}L\cos\theta + L^2\dot{\theta}^2\cos^2\theta + L^2\dot{\theta}^2\sin^2\theta\right) + \frac{1}{2}m\dot{x}^2$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \dot{x}\dot{\theta}ML\cos\theta + \frac{1}{2}ML^2\dot{\theta}^2$$

$$x' = L\sin\theta$$
$$\rightarrow \dot{x}' = L\dot{\theta}\cos\theta$$
$$y' = L\cos\theta$$
$$\rightarrow \dot{y}' = -L\dot{\theta}\sin\theta$$

$$U = -mgL\cos\theta$$

$$\Rightarrow L = T - U$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \dot{x}\dot{\theta}ML\cos\theta + \frac{1}{2}ML^2\dot{\theta}^2 + MgL\cos\theta$$

equations of motion:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Rightarrow 0 = \frac{d}{dt}\left[(M+m)\dot{x} + \dot{\theta}mL\cos\theta\right]$$

$$= (M+m)\ddot{x} + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta \quad (1)$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Rightarrow -ML\dot{x}\dot{\theta}\sin\theta - MgL\sin\theta = \frac{d}{dt}\left[\dot{x}LM\cos\theta + ML^2\dot{\theta}\right]$$

$$= ML\ddot{x}\cos\theta - ML\dot{x}\dot{\theta}\sin\theta + ML^2\ddot{\theta}$$

(2)

For small oscillations about $\theta = 0$, use $\sin\theta \approx \theta$, $\cos\theta \approx 1$ and neglect terms of quadratic order or higher in $x, \theta, \dot{x}, \dot{\theta}$:

$$\begin{cases} (M+m)\ddot{x} + ML\ddot{\theta} = 0 \\ ML\ddot{x} + ML^2\ddot{\theta} = -MgL\theta \end{cases}$$

or:
$$\underbrace{\begin{bmatrix} M+m & ML \\ ML & ML^2 \end{bmatrix}}_{\underline{M}} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\underline{x}}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -MgL \end{bmatrix}}_{\underline{K}} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{\underline{x}}$$

assume $\underline{x} = \underline{z} e^{i\omega t} \Rightarrow \left(K + \omega^2 M\right)\underline{z} = \underline{0}$

where $K + \omega^2 M = \begin{bmatrix} \omega^2(M+m) & \omega^2 ML \\ \omega^2 ML & \omega^2 ML^2 - MgL \end{bmatrix}$

thus $0 = \det\left(K + \omega^2 M\right) = \omega^4 (M+m)ML^2 - \omega^2 (M+m)MgL - \omega^4 M^2 L^2$

$$= \omega^2\left[\omega^2(M+m)L - (M+m)g - \omega^2 ML\right]ML$$

$$= \omega^2\left[\omega^2 mL - (M+m)g\right]ML$$

$\Rightarrow \omega^2 = 0$ or $\omega^2 = \frac{M+m}{m} \cdot \frac{g}{L} \Rightarrow \omega^{\cancel{2}} = \sqrt{\left(1 + \frac{M}{m}\right)\frac{g}{L}}$

Normal modes:

case $\omega=0$: $K+\omega^2 M = \begin{bmatrix} 0 & 0 \\ 0 & -MgL \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \underline{z} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

If $\underline{x}(t) = \underline{z} f(t)$ then $M\ddot{\underline{x}} = K\underline{x} \Rightarrow M\underline{z}f'' = \underbrace{K\underline{z}}_{0} f$

$$\Rightarrow f'' = 0$$

$$\Rightarrow \underline{x}(t) = \underline{z}(At+B)$$

$$\Rightarrow \begin{cases} x(t) = At+B \\ \theta(t) = 0 \end{cases}$$ (cart rolls at const. speed with pendulum vertical)

case $\omega^2 = \frac{M+m}{m} \cdot \frac{g}{L}$:

$$K+\omega^2 M = \begin{bmatrix} \frac{(M+m)^2}{m}\frac{g}{L} & \frac{M+m}{m}\cdot Mg \\ \frac{M+m}{m}\cdot Mg & \frac{M+m}{m} MLg - MgL \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(M+m)^2}{m}\frac{g}{L} & \frac{M+m}{m}\cdot Mg \\ \frac{M+m}{m}\cdot Mg & \frac{M}{m}\cdot MgL \end{bmatrix}$$

$$\xrightarrow{R_1 \times ML} \begin{bmatrix} \frac{(M+m)^2}{m} Mg & \frac{M+m}{m}\cdot M^2 gL \\ \frac{M+m}{m} Mg & \frac{1}{m} M^2 gL \end{bmatrix}$$

$$R_2 - (M+m)R_1 \atop R_1 \cdot \frac{m}{M\cdot m} \cdot \frac{1}{g} \longrightarrow \begin{bmatrix} \frac{M+m}{L} & M \\ 0 & 0 \end{bmatrix} \longrightarrow \frac{M+m}{L} z_1 = -M z_2$$

So $z_1 = -L$ is a nice choice:

$$\Rightarrow \underline{z} = \begin{pmatrix} -L \\ 1 + \frac{m}{M} \end{pmatrix}$$

$\Rightarrow$ motion is out-of-phase; L controls amplitude of x; mass ratio controls amplitude of θ.