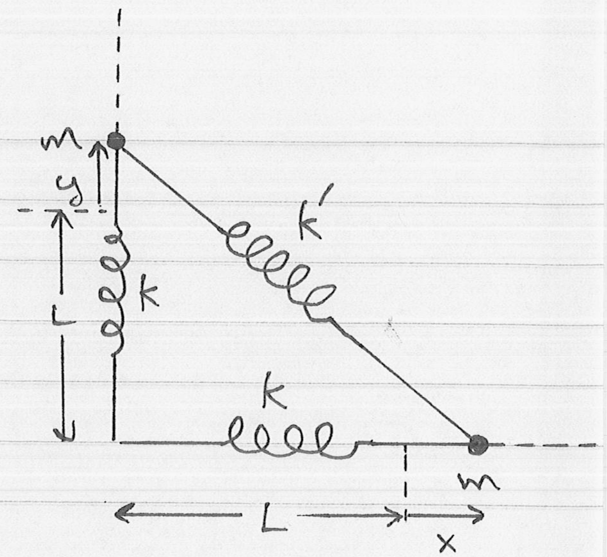
11.18

Measure x, y relative to rest lengths of springs.



Displacement of spring k' is:

$$\sqrt{(L+x)^2+(L+y)^2} - \underbrace{\sqrt{2}L}_{\text{rest length when } x=y=0.}$$

$$\text{So} \quad U = \tfrac{1}{2}kx^2 + \tfrac{1}{2}ky^2 + \tfrac{1}{2}k'\left(\sqrt{(L+x)^2+(L+y)^2} - \sqrt{2}L\right)^2$$

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

$$\Rightarrow L = T - U$$

$$= \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) - \tfrac{1}{2}k(x^2+y^2) - \tfrac{1}{2}k'\Big(\underbrace{\sqrt{(L+x)^2+(L+y)^2} - \sqrt{2}L}_{f(x,y)}\Big)^2$$

Equations of motion:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Rightarrow m\ddot{x} = -\cancel{kx} - \cancel{ky} \overbrace{- k' f(x,y)\frac{\partial f}{\partial x}}^{F_x}$$

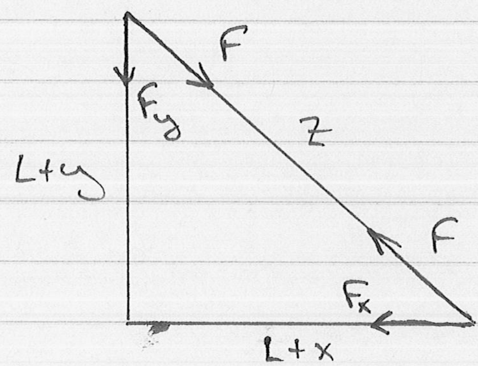
$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Rightarrow m\ddot{y} = -ky \underbrace{- k' f(x,y)\frac{\partial f}{\partial y}}_{F_y}$$

Linearizing about $x=y=0$ gives

$$\left.\begin{aligned} f(x,y)\frac{\partial f}{\partial x} &\approx \tfrac{1}{2}(x+y) + \cdots \\ f(x,y)\frac{\partial f}{\partial y} &\approx \tfrac{1}{2}(x+y) + \cdots \end{aligned}\right\}$$ e.g. by Maple, Wolfram Alpha, …

(or see pg. following)

Here is one approach to linearizing:



$$z^2 = (L+x)^2 + (L+y)^2 \Rightarrow 2z\,dz = 2(L+x)dx + 2(L+y)dy$$

$$x \approx y \approx 0 \Rightarrow z = \sqrt{2}L \Rightarrow \sqrt{2}\,dz = dx + dy$$

The spring force is $F = k' \cdot dz = \frac{k'}{\sqrt{2}}(dx + dy)$.

$$\therefore \quad F_x = \frac{-L}{z} \cdot F = \frac{-1}{\sqrt{2}} F = \frac{-k'}{2}(dx + dy)$$

and similarly,

$$F_y = \frac{-L}{z} F = -\frac{k'}{2}(dx + dy)$$

So for small oscillations (about $x=y=0$) $F_x = F_y = -\frac{k'}{2}(x+y)$.

so the linearized equations (for small oscillations) are:

$$m\ddot{x} = -kx - \tfrac{1}{2}k'(x+y)$$
$$m\ddot{y} = -ky - \tfrac{1}{2}k'(x+y)$$

$$\Rightarrow \underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix}}_{\ddot{\underline{x}}} = \underbrace{\begin{bmatrix} -k-\tfrac{1}{2}k' & -\tfrac{1}{2}k' \\ -\tfrac{1}{2}k' & -k-\tfrac{1}{2}k' \end{bmatrix}}_{K} \begin{bmatrix} x \\ y \end{bmatrix}$$

Assume $\underline{x}(t) = \underline{z}e^{i\omega t} \Rightarrow (K + \omega^2 M)\underline{z} = \underline{0}$

with $K + \omega^2 M = \begin{bmatrix} \omega^2 m - k - \tfrac{1}{2}k' & -\tfrac{1}{2}k' \\ -\tfrac{1}{2}k' & \omega^2 m - k - \tfrac{1}{2}k' \end{bmatrix}$

Non-trivial solutions require:

$$0 = \det(K + \omega^2 M) = (\omega^2 m - k - \tfrac{1}{2}k')^2 - (\tfrac{1}{2}k')^2$$

$$\Rightarrow \omega^2 m - k - \tfrac{1}{2}k' = \pm \tfrac{1}{2}k'$$

$$\Rightarrow \omega^2 m = k \text{ or } k + k'$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{k}{m}} \text{ or } \sqrt{\frac{k+k'}{m}}}$$

Solve for modes...

case $\omega^2 = k/m$:

$$\underline{0} = (K + \tfrac{k}{m}M)\underline{z} = \begin{pmatrix} -\tfrac{1}{2}k' & -\tfrac{1}{2}k' \\ -\tfrac{1}{2}k' & -\tfrac{1}{2}k' \end{pmatrix}\underline{z} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \boxed{\underline{z} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

- ω indep of k' because k' remains at rest length (out-of-phase) case $\omega^2 = \dfrac{k+k'}{m}$:

$$\underline{0} = \left(K + \frac{k+k'}{m} M\right)\underline{z} = \begin{bmatrix} \frac{1}{2}k' & -\frac{1}{2}k' \\ -\frac{1}{2}k' & \frac{1}{2}k' \end{bmatrix}\underline{z} \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \boxed{\underline{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Motion is in-phase ... higher frequency since k' adds stiffness.