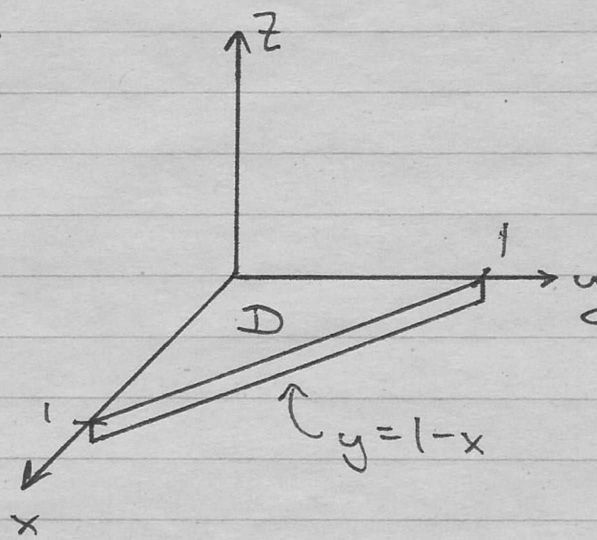
10.37 Thin, uniform triangle as shown.
Mass density $\sigma = 24$.



a) $I_{xx} = \iint (y^2 + z^2)\,dm$

$= \iint_D y^2 \sigma \,dx\,dy$

$= 24 \underbrace{\int_0^1 \int_0^{1-x} y^2 \,dy\,dx}_{1/12}$

$= 2.$ ($= I_{yy}$ by symmetry)

$I_{zz} = \iint (x^2+y^2)\,dm$

$= \iint_D (x^2+y^2)\,\sigma\,dx\,dy = 24 \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx = 4.$

$I_{xy} = -\iint xy\,dm$

$= -\iint_D xy\,\sigma\,dx\,dy = -24 \int_0^1 \int_0^{1-x} xy\,dy\,dx = -1$

$I_{xz} = -\iint xz\,dm = \iint_D 0\,dx\,dy = 0$

$I_{yz} = -\iint yz\,dm = \iint_D 0\,dx\,dy = 0$

$$\therefore \quad I = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

b) Principal axes (eigenvectors $\vec{v}$) and corresponding moments (eigenvalues λ):

$$0 = \det(I - \lambda \mathbb{1}) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$= \left[(2-\lambda)^2 - 1 \right] (4-\lambda)$$

$$\Rightarrow \lambda = 4 \text{ or } \lambda = 2 \pm 1 = 1 \text{ or } 3.$$

<u>case $\lambda = 1$:</u>

$$I - \lambda \mathbb{1} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(I - \lambda \mathbb{1}) \vec{v} = \vec{0} \Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ (up to a multiplicative constant)

case $\lambda = 3$:

$$I - \lambda \mathbb{1} = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

case $\lambda = 4$:

$$I - \lambda \mathbb{1} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So the principal axes and moments are:

$\lambda_1 = 1 \qquad \vec{v}_1 = (1, 1, 0)$

$\lambda_2 = 3 \qquad \vec{v}_2 = (1, -1, 0)$

$\lambda_3 = 4 \qquad \vec{v}_3 = (0, 0, 1)$