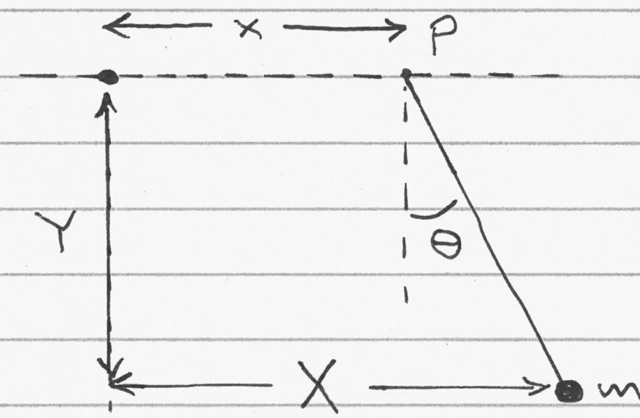
(a) $X = x + l\sin\theta$
$Y = l\cos\theta$



$\rightarrow \dot{X} = \dot{x} + l\dot{\theta}\cos\theta$
$\dot{Y} = -l\dot{\theta}\sin\theta$

kinetic energy:

$$T = \frac{1}{2}m\left(\dot{X}^2 + \dot{Y}^2\right)$$

$$= \frac{1}{2}m\left(\dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta\right)$$

$$= \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2$$

potential energy:

$$U = -mgY = -mgl\cos\theta$$

Lagrangian:

$$L = T - U$$

$$= \frac{1}{2}m\dot{x}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

(b) $L_x = \frac{d}{dt} L_{\dot{x}}$

$$= m\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$$

$$\Rightarrow \underbrace{0}_{\text{total force in x-direction}} = \frac{d}{dt}\Big(\underbrace{m\dot{x} + ml\dot{\theta}\cos\theta}_{\text{linear momentum in x-direction}}\Big) \qquad (1)$$

$L_\theta = \frac{d}{dt} L_{\dot{\theta}}$

$$\Rightarrow \underbrace{-ml\dot{x}\dot{\theta}\sin\theta}_{\text{torque due to...? fictitious torque due to non-inertial coord. system}} \underbrace{- mgl\sin\theta}_{\text{torque due to gravity}} = \frac{d}{dt}\Big(\underbrace{ml\dot{x}\cos\theta + ml^2\dot{\theta}}_{\text{angular momentum}}\Big)$$

$$\Rightarrow -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta = ml\ddot{x}\cos\theta - ml\dot{x}\dot{\theta}\sin\theta + ml^2\ddot{\theta}$$

$$\Rightarrow -mgl\sin\theta = ml\ddot{x}\cos\theta + ml^2\ddot{\theta}$$

$$\Rightarrow -\cancel{\tfrac{g}{l}\sin\theta} = -g\sin\theta = \ddot{x}\cos\theta + l\ddot{\theta} \qquad (2)$$

After simplifying:

$$\begin{cases} \ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta = 0 & (1) \\ \ddot{x}\cos\theta + l\ddot{\theta} = -g\sin\theta & (2) \end{cases}$$

1c) Because $L_x = 0$, the quantity $L_{\dot{x}}$ (conjugate momentum relative to x, i.e. horizontal component of linear momentum) is conserved.

In terms of Newtonian mechanics, this is because there is no horizontal external force.

1d) small angle approx: $\sin\theta \approx \theta$, $\cos\theta \approx 1$

$$\Rightarrow \quad 0 = m\ddot{x} + ml\ddot{\theta} \qquad (1)$$

$$-mgl\theta = ml\ddot{x} + ml^2\ddot{\theta} \qquad (2)$$

$$\Rightarrow \quad \ddot{x} = -l\ddot{\theta} \qquad (1)$$

$$-g\theta = \ddot{x} + l\ddot{\theta} \qquad (2)$$

$$(2) \Rightarrow \; -g\theta = -l\ddot{\theta} + l\ddot{\theta} = 0 \qquad (!?)$$

What happened here? There is an exact cancellation of linear-order terms, due to having no mass at point P. Unfortunately, linearization doesn't give any useful information about this system!

Going back to the exact equations:

$$(1) \Rightarrow \ddot{x} = l\dot{\theta}^2 \sin\theta - l\ddot{\theta}\cos\theta$$

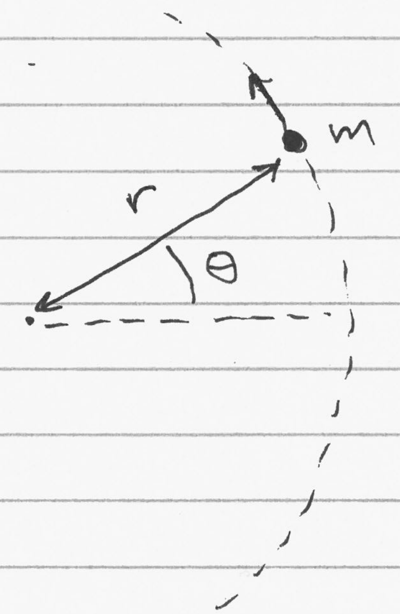
Substitute this into (2):

$$\left(l\dot{\theta}^2 \sin\theta - l\ddot{\theta}\cos\theta\right)\cos\theta + l\ddot{\theta} = -g\sin\theta$$

$$\Rightarrow \dot{\theta}^2 \sin\theta + \underbrace{\left(1-\cos^2\theta\right)}_{\sin^2\theta}\ddot{\theta} = -\frac{g}{l}\sin\theta$$

$$\Rightarrow \boxed{\dot{\theta}^2 + \ddot{\theta}\sin\theta = -\frac{g}{l}}$$

This DE is not only non-linear, but has a singular point at $\theta = 0$. It can't be linearized because leading-order terms are already quadratic in θ!



2) kinetic energy:

$$T = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 \right)$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

potential energy:

$$U = 0$$

Lagrangian:

$$L = T - U$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$$

constraint: $f(r,\theta) = r = R$

use Lagrange multiplier in eqs. of motion:

$$\begin{cases} L_\theta + \lambda f_\theta = \frac{d}{dt} L_{\dot\theta} \rightarrow 0 + 0 = \frac{d}{dt}\left(m r^2 \dot{\theta} \right) & (1) \\ L_r + \lambda f_r = \frac{d}{dt} L_{\dot r} \rightarrow m r \dot{\theta}^2 + \lambda = \frac{d}{dt}\left(m\dot{r} \right) & (2) \end{cases}$$

radial momentum

(radial) constraint force due to wire

substitute $r = R$, $\dot{r} = 0$, $\dot{\theta} = \omega$:

$$(2) \Rightarrow mR\omega^2 + \lambda = 0$$

$$\Rightarrow \lambda = -mR\omega^2$$

= radial force of wire on bead