THOMPSON RIVERS UNIVERSITY

PHYS 3200
Advanced Mechanics

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MIDTERM EXAM
SOLUTIONS

2 March 2012 14:30–15:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		20
3		14
TOTAL:		44

/10

Problem 1: Find the function $y(x)$ with $y(1) = 1$, $y(2) = 7$ that minimizes the value of the following integral:

$$I[y] = \int_1^2 \frac{y'(x)^3}{x^2}\, dx$$

We have $I[y] = \int_1^2 F(x, y, y')\, dx$ with

$$F(x, y, y') = y'^3/x^2.$$

So the optimal function y must satisfy the Euler-Lagrange equation:

$$\frac{d}{dx} F_{y'} = F_y \implies \frac{d}{dx}\frac{3y'^2}{x^2} = 0$$

This has an easy first integral:

$$\frac{3y'^2}{x^2} = C_1$$

$$\implies y' = \pm\frac{\sqrt{C_1}}{\sqrt{3}} x = B_1 x \quad (B_1 \in \mathbb{R})$$

Integrating this gives:

$$\implies y = \frac{1}{2} B_1 x^2 + B_2 \quad (B_1, B_2 \in \mathbb{R})$$

Imposing the boundary conditions gives:

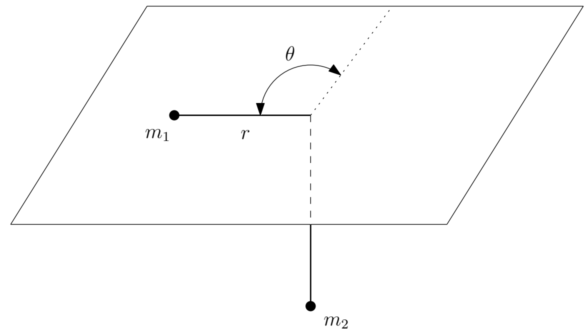
$$\begin{cases} 1 = y(1) = B_1/2 + B_2 \\ 7 = y(2) = 2B_1 + B_2 \end{cases} \implies (7-1) = (2-1/2)B_1 \implies B_1 = 4 \implies B_2 = 1 - (4)/2 = -1$$

Therefore,

$$\boxed{y(x) = 2x^2 - 1}$$

/20 **Problem 2:** A mass m_1 moves on a horizontal frictionless table. Attached to m_1 is a taut string that hangs vertically through a small hole in the table, through which the string can pass frictionlessly. A second mass m_2 is attached to the lower end of the string under the table. (See the diagram below).

/3 (a) Write the Lagrangian for this system in terms of the coordinates r, θ.



$$T = \tfrac{1}{2}m_1\left(\dot{r}^2 + (r\dot{\theta})^2\right) + \tfrac{1}{2}m_2\dot{r}^2 \qquad U = m_2 g r \implies \boxed{L = T - U = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2}m_1 r^2\dot{\theta}^2 - m_2 g r}$$

/6 (b) Determine the equations of motion for r, θ. Comment on the special case when $\dot{\theta} = 0$.

$$\frac{d}{dt}L_{\dot{r}} = L_r \implies \boxed{(m_1+m_2)\ddot{r} = m_1 r\dot{\theta}^2 - m_2 g}$$

$$\frac{d}{dt}L_{\dot{\theta}} = L_\theta \implies \boxed{\frac{d}{dt}(m_1 r^2\dot{\theta}) = 0}$$

In the case $\dot{\theta} = 0$ the first equation reduces to

$$(m_1+m_2)\ddot{r} = -m_2 g$$

which is exactly what we should expect from Newton's 2nd law: a constant gravitational force $m_2 g$ accelerating the total mass $m_1 + m_2$.

/3 (c) Show that the equation of motion in θ has a first integral $m_1 r^2\dot{\theta} = C$. Hence eliminate θ to get

$$(m_1+m_2)\ddot{r} = \frac{C^2}{m_1 r^3} - m_2 g$$

The equation $\frac{d}{dt}(m_1 r^2\dot{\theta}) = 0$ gives an easy first integral:

$$m_1 r^2\dot{\theta} = C \quad \text{(angular momentum)} \implies \dot{\theta} = \frac{C}{m_1 r^2}$$

Substituting this into the DE for r gives

$$(m_1+m_2)\ddot{r} = m_1 r\left(\frac{C}{m_1 r^2}\right)^2 - m_2 g = \frac{C^2}{m_1 r^3} - m_2 g$$

(d) Find $\dot{\theta}$ such that $r = r_0$ is constant (i.e. m_2 is in equilibrium while m_1 moves in a circle at constant speed).

/3

If r is constant then $\ddot{r} = \dot{r} = 0$ and the DE in r from part (b) gives

$$0 = m_1 r_0 \dot{\theta}^2 - m_2 g \implies \boxed{\dot{\theta} = \sqrt{\frac{m_2}{m_1}}\sqrt{\frac{g}{r_0}}}$$

(e) Find the frequency of small oscillations about $r = r_0$ in part (d).

/5

From the DE in part (c) the equilibrium position ($\ddot{r} = 0$) is given by

$$0 = \frac{C^2}{m_1 r_0^3} - m_2 g \implies C^2 = m_1 m_2 g r_0^3$$

Using this to rewrite the DE in terms of r_0 gives

$$\begin{aligned}(m_1 + m_2)\ddot{r} &= \frac{C^2}{m_1 r^3} - m_2 g \\ &= \frac{m_1 m_2 g r_0^3}{m_1 r^3} - m_2 g \\ &= m_2 g \underbrace{\left(\frac{r_0^3}{r^3} - 1\right)}_{f(r)}\end{aligned}$$

We can linearize this by finding he Taylor series for $f(r)$ about $r = r_0$:

$$\begin{cases} f(r_0) = 0 \\ f'(r_0) = -3\frac{r_0^3}{r_0^4} = -\frac{3}{r_0} \end{cases} \implies f(r) \approx -\frac{3}{r_0}(r - r_0) \quad \text{for small } r - r_0$$

With $r = r_0 + \varepsilon$ the linearized DE becomes

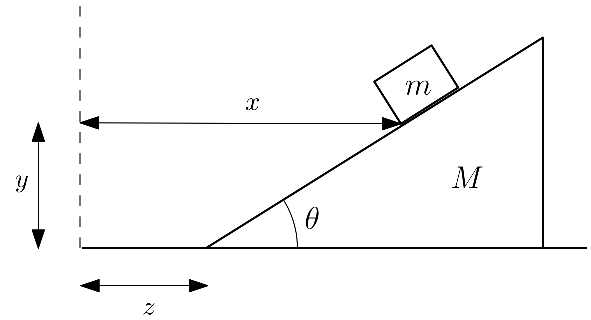
$$(m_1 + m_2)\ddot{\varepsilon} = -m_2 g \frac{3}{r_0}\varepsilon \implies \ddot{\varepsilon} + \underbrace{\left(\frac{m_2}{m_1 + m_2}\frac{3g}{r_0}\right)}_{\omega^2}\varepsilon = 0$$

so the frequency ω of small oscillations is

$$\implies \boxed{\omega = \sqrt{\frac{m_2}{m_1 + m_2}}\sqrt{\frac{3g}{r_0}}}$$

/14 **Problem 3:** A block of mass m slides without friction down a wedge of mass M and angle θ, which itself slides frictionlessly on a horizontal surface (see diagram below). The block has displacement vector (x, y) and the wedge has horizontal displacement z relative to the cartesian coordinate system shown.

/3 (a) Write the Lagrangian for this system in terms of the given coordinates x, y, z.



$$\begin{cases} T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}M\dot{z}^2 \\ U = Mgy \end{cases} \implies \boxed{L = T - U = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}M\dot{z}^2 - Mgy}$$

/2 (b) Find a constraint relating x, y, z such that the block remains in contact with the wedge.

$$\tan\theta = \frac{y}{x-z} \implies \boxed{\underbrace{y + (z-x)\tan\theta}_{f(x,y,z)} = 0}$$

/6 (c) Use the method of Lagrange multipliers to find the wedge's acceleration $\ddot{z}$.

$$\frac{d}{dt}L_{\dot{x}} = L_x + \lambda f_x \implies m\ddot{x} = -\lambda\tan\theta \tag{1}$$

$$\frac{d}{dt}L_{\dot{y}} = L_y + \lambda f_y \implies m\ddot{y} = -Mg + \lambda \tag{2}$$

$$\frac{d}{dt}L_{\dot{z}} = L_z + \lambda f_z \implies M\ddot{z} = \lambda\tan\theta \tag{3}$$

The constraint equation also gives

$$\ddot{y} + (\ddot{z}-\ddot{x})\tan\theta = 0. \tag{4}$$

Substiting equations (1)–(3) into (4) gives

$$\frac{-Mg+\lambda}{m} + \left(\frac{\lambda}{M}\tan\theta + \frac{\lambda}{m}\tan\theta\right)\tan\theta = 0 \implies \lambda\left[\frac{1}{m} + \left(\frac{1}{M}+\frac{1}{m}\right)\tan^2\theta\right] = \frac{M}{m}g$$

$$\implies \lambda = \frac{\frac{M}{m}g}{\frac{1}{m} + \left(\frac{1}{M}+\frac{1}{m}\right)\tan^2\theta} = \frac{Mg}{1+\left(\frac{m}{M}+1\right)\tan^2\theta}$$

$$\implies \boxed{\ddot{z} = \frac{\lambda}{M}\tan\theta = \frac{g}{1+\left(\frac{m}{M}+1\right)\tan^2\theta}\tan\theta}$$

/3 (d) Use your Lagrange multiplier to find the horizontal component of force that the block exerts on the wedge.

From the z equation of motion it is clear that the term $\lambda\tan\theta$ is the z-component of the contact force. Thus

$$\boxed{F_z = \lambda\tan\theta = \frac{Mg}{1+\left(\frac{m}{M}+1\right)\tan^2\theta}\tan\theta}$$