PHYS 3200 Lecture Notes

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1 Introduction

Reading mathematics and physics is a fairly advanced skill. Most students find that an interactive lecture format is an easier way to get acquainted with new ideas. But reading technical material is a skill worth learning, and this seems as good a time as any to start working on it.

The dates indicated here are the days I would have covered the material in class, and I've put them here to help you pace your work. In the right-hand margin I have indicated the relevant sections of the textbook. In some places I have suggested specific sections you should definitely read, and exercises you should attempt as you progress. I will add to this document as we go, so **check back here often**.

I've purposefully made these notes brief and to the point. You should use the textbook for supplemental information, and especially be sure to do the assigned readings. I aim to get you solving relevant problems on your own as quickly as possible. If you read these notes, do the assigned readings, and still find it difficult to get started on the problems, send me an email ASAP. I will help however I can, and your feedback will help me to make these notes better.

March 24

2 Rotation of a Rigid Object

2.1 Wobbling Motion

We showed in class that for a rigid body spinning with angular velocity $\boldsymbol{\omega}$ about an axis through a given point O, the angular momentum vector $\mathbf{L}$ is

$$\mathbf{L} = I\boldsymbol{\omega} \tag{1}$$

where I is the moment of inertia tensor, a 3×3 matrix. All of the quantities here are expressed in the body frame, i.e. a rotating coordinate system anchored to the body.

One consequence of eq. (1) is that, in general, $\mathbf{L}$ and $\boldsymbol{\omega}$ are *not* parallel: multiplying $\boldsymbol{\omega}$ by I effects a linear transformation (scaling and rotation) that results in $\mathbf{L}$ being oriented differently from $\boldsymbol{\omega}$.

This has consequences for the motion of the object. In the absence of an external torque, $\mathbf{L}$ is constant. But the geometric relationship between $\mathbf{L}$ and $\boldsymbol{\omega}$ is fixed by eq. (1), so as the object spins about $\boldsymbol{\omega}$ the vector $\mathbf{L}$ must be changing (carried around by the body's rotation) unless $\boldsymbol{\omega}$ itself is changing. Thus, in the absence of an external torque, the axis of rotation must be constantly changing. This causes a wobbling/tumbling motion. We will describe this wobble more precisely when we come to Euler's equations.

Reading Assignment: In the textbook, read from the bottom third of p. 374 to the end of p. 375.

2.2 Principal Axes

10.4, 10.5

So the question naturally arises: "is there an axis $\boldsymbol{\omega}$ about which an object can spin without wobbling in the absence of an external torque?" From the discussion above, this requires that $\mathbf{L}$ and $\boldsymbol{\omega}$ be parallel (Why? In the absence of an external torque, $\mathbf{L}$ constant. But then so is $\boldsymbol{\omega}$, if $\boldsymbol{\omega}$ if parallel to $\mathbf{L}$.) Mathematically this means

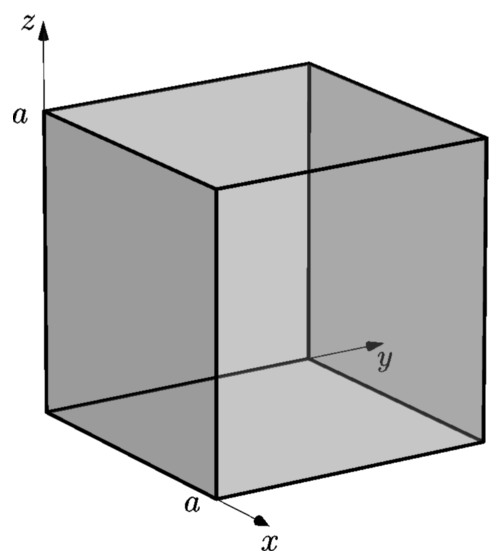
$$\mathbf{L} = \lambda\boldsymbol{\omega} \tag{2}$$

for some (non-zero) scalar λ. Substituting eq. (1) into this gives

$$I\boldsymbol{\omega} = \lambda\boldsymbol{\omega}. \tag{3}$$

This mean that ω is an eigenvector of the matrix I, with corresponding eigenvalue λ.

Example: Find the principal axes for a uniform solid cube of side length a, rotating about an axis through one corner.



Solution: We already showed that for rotation about an axis through the origin,

$$I = \frac{Ma^2}{12}\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}. \tag{4}$$

Finding the principal axes (eigenvectors of I) is an exercise in linear algebra. As usual, we need to find the eigenvalues first. We have

$$I\boldsymbol{\omega} = \lambda\boldsymbol{\omega} \iff I\boldsymbol{\omega} - \lambda\boldsymbol{\omega} = \mathbf{0} \iff (I - \lambda\mathbf{1})\boldsymbol{\omega} = \mathbf{0}. \tag{5}$$

Here we've used **1** to represent the 3×3 identity matrix (annoyingly, I is already used for the moment of inertia tensor). We seek non-trivial (i.e. non-zero) vectors $\boldsymbol{\omega}$ that satisfy eq. (5). This requires that the matrix $I - \lambda\mathbf{1}$ be singular (**why?**). That is (with $k = Ma^2/12$ to simplify things):

$$0 = \det(I - \lambda\mathbf{1}) = \det\left(k\begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 8k-\lambda & -3k & -3k \\ -3k & 8k-\lambda & -3k \\ -3k & -3k & 8k-\lambda \end{vmatrix}. \tag{6}$$

Expanding the 3×3 determinant and factoring (check this!) gives the equation

$$(2k - \lambda)(11k - \lambda)^2 = 0 \tag{7}$$

so we get three roots (counting multiplicity):

$$\lambda_1 = 2k = \frac{Ma^2}{6}, \qquad \lambda_2 = \lambda_3 = 11k = \tfrac{11}{12}Ma^2. \tag{8}$$

To find the eigenvectors (the principal axes) we substitute each root in turn into eq. (5) and row reduce to solve for $\boldsymbol{\omega}$.

<u>case $\lambda = 2k$:</u>

$$I - \lambda\mathbf{1} = k\begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Interpreting this as the coefficient matrix in eq. (5) and writing $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$, we see that ω_z is a free variable and that $\omega_x = \omega_y = \omega_z$. As always, the eignvector is only determined up to a multiplicative constant, so we might as well say that $\boldsymbol{\omega} = (1, 1, 1)$. However, it will be convenient later if we convert this to a unit vector

$$\mathbf{e}_1 = \tfrac{1}{\sqrt{3}}(1, 1, 1) \tag{10}$$

that gives the direction of the angular velocity vector.

<u>case $\lambda = 11k$:</u>

$$I - \lambda\mathbf{1} = k\begin{bmatrix} -3 & -3 & -3 \\ -3 & -3 & -3 \\ -3 & -3 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{11}$$

Now we have *two* free variables; the solution set is a plane through the origin. Any two linearly independent vectors in this plane will serve as eigenvectors; the choice is arbitrary. One approach is to write the general solution as

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -\omega_y - \omega_z \\ \omega_y \\ \omega_z \end{bmatrix} = \omega_y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \omega_z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{12}$$

This shows that any eigenvector must be a linear combination of $(-1, 1, 0)$ and $(-1, 0, 1)$. Normalizing these gives the directions

$$\mathbf{e}_2 = \tfrac{1}{\sqrt{2}}(-1, 1, 0) \quad \text{and} \quad \mathbf{e}_3 = \tfrac{1}{\sqrt{2}}(-1, 0, 1). \tag{13}$$

Reading Assignment: Read Sec. 10.4.

Exercises: 10.35, 10.36, 10.37

2.3 Some Observations

Notice in the example above that there are three principal axes, and that these are mutually perpendicular. In fact this is the case for *any* rigid object: from linear algebra you might already know that any real-valued symmetric matrix (like the moment of inertia tensor) has a set of three mutually orthogonal eigenvectors.

An object's principal axes are generally aligned with its axes of symmetry, if there are any. In the example above the principal axis $\mathbf{e}_1$ is oriented from O to opposite corner. In the absence of an external torque, the cube can spin about this axis without wobbling.

The eigenvalues have a physical interpretation as well. For an object spinning about a principal axis, eq. (2) gives the angular momentum $\mathbf{L} = I\boldsymbol{\omega} = \lambda\boldsymbol{\omega}$. That is, λ is just the moment of inertia for rotation about this axis.

In calculating the moment of inertia tensor for some object, we are free to choose the coordinate system (we only require that the origin lie on the axis of rotation). In each coordinate system, the moment of inertia tensor will be different. Among all coordinate systems, the most convenient is one where the axes coincide with the principal axes $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for the object. In this case, the moment of inertia tensor becomes simply

$$I = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{14}$$

This is a diagonal matrix, and we say that I is *diagonalized* in this coordinate system.

If we identify the x-axis with $\mathbf{e}_1$, the y-axis with $\mathbf{e}_2$ and the z-axis with $\mathbf{e}_3$, then the angular momentum vector is simply:

$$\mathbf{L} = I\boldsymbol{\omega} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \lambda_1\omega_x \\ \lambda_2\omega_y \\ \lambda_3\omega_z \end{bmatrix}. \tag{15}$$

This is a big simplification. For this reason, in dealing with rigid-body motion it is usually preferable to work in a coordinate system aligned with the principal axes.

March 25

2.4 Precession of a Spinning Top

10.6

A spinning top that is not perfectly vertical with exhibit precession: the orientation of the top will gradually rotate about the vertical axis. If you have access to such a thing, play around with this. There is a nice demonstration here (skip to the middle after watching the intro):

`https://www.youtube.com/watch?v=SWyuvAdxtKs`

The cause of precession is the torque $\boldsymbol{\tau}$ (acting about the point of contact with the table or ground) due to gravity. This torque causes a change in the angular momentum of the top since $\dot{\mathbf{L}} = \boldsymbol{\tau}$. But because $\boldsymbol{\tau} = \dot{\mathbf{L}}$ is perpendicular to $\mathbf{L}$, the torque causes $\mathbf{L}$ to change in direction but not magnitude (recall Problem 1.45). A good hand-waving demonstration of this can be seen here:

`https://www.youtube.com/watch?v=ty9QSiVC2g0`

As promised, the equations of motion for the top are easiest to describe if we choose a coordinate system that coincides with the principal axes of the top. Have a look at Fig. 10.7. Here the unit vector $\mathbf{e}_3$ is one of the top's principal axes (the main axis of symmetry, assuming the top has cylindrical symmetry as most tops do). If the top has angular velocity $\boldsymbol{\omega} = \omega \mathbf{e}_3$ then the angular momentum is $\mathbf{L} = I\boldsymbol{\omega} = \lambda_3 \boldsymbol{\omega} = \lambda_3 \omega \mathbf{e}_3$ (this formula is so simple because the rotation is about a principal axis).

In the absence of gravity (and friction), the top would remain spinning with constant orientation: $\dot{\mathbf{L}} = 0 \implies \dot{\boldsymbol{\omega}} = 0$. This is what we observe if the top is aligned vertically.

If the top is not vertical, the force of gravity $M\mathbf{g}$ causes a torque

$$\boldsymbol{\tau} = \mathbf{R} \times (M\mathbf{g}) \tag{16}$$

where $\mathbf{R}$ is the vector from the contact point with the floor/table to the top's center of gravity (through which the effective force of gravity acts). With careful attention to Fig. 10.7, the right-hand rule will help you discover that $\boldsymbol{\tau}$ is horizontal and perpendicular to $\boldsymbol{\omega}$ (into the page). Since $\boldsymbol{\tau} = \dot{\mathbf{L}}$ this tells you what direction $\mathbf{L}$ will move.

A little more algebra shows that

$$\dot{\mathbf{e}}_3 = \boldsymbol{\Omega} \times \mathbf{e}_3 \tag{17}$$

where

$$\boldsymbol{\Omega} = \frac{MgR}{\lambda_3 \omega} \hat{\mathbf{z}}. \tag{18}$$

This describes a rotation of $\mathbf{e}_3$ about the vertical $\hat{\mathbf{z}}$ axis (see eq. (9.30)) with angular frequency $\frac{MgR}{\lambda_3 \omega}$.

Note that the rate of precession is independent of the off-vertical angle θ, a fact that you can observe in the videos above. Real-world spinning tops behave slightly differently, mainly due to the action of friction which adds an additional torque. If you think carefully about this you should be able to figure out the direction of this torque and thus the effect on the motion of the top.

Earth itself acts as a spinning top. The small gravitational torque due to the gravity of the sun and moon acting on Earth's "equatorial bulge" (slight deviation from sphericity) causes Earth's axis to precess. The period of this precession is about $26,000$ years. Here is an excellent video about that:

`https://www.youtube.com/watch?v=adzx547ptck`

Reading Assignment: Sec. 10.6

Exercise: 10.39

March 31

2.5 Euler's Equations

10.7

We are now in a position to describe quantitatively the tumbling motion of a rigid body about an axis that is not a principal axis.

Recall that the rotational analog of Newton's 2nd Law is

$$\dot{\mathbf{L}} = \boldsymbol{\tau} \tag{19}$$

where $\mathbf{L}$ is the total angular momentum of a rigid body and $\boldsymbol{\tau}$ is the applied torque. However, eq. (19) applies in an inertial frame S_0, whereas our description of the angular momentum using the moment of inertia tensor was done in the *body frame* S (i.e. the coordinate system embedded in and rotating with the rigid body). To apply eq. (19) we need to translate from the inertial frame S_0 to the rotating frame S.

Recall that for a coordinate system rotating with angular velocity vector $\boldsymbol{\omega}$ we have, for *any* quantity $\mathbf{q}$ measured simultaneously in both frames, that

$$\left(\frac{d\mathbf{q}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{q}}{dt}\right)_S + \boldsymbol{\omega} \times \mathbf{q}. \tag{20}$$

Applying this to the angular momentum vector gives

$$\boldsymbol{\tau} = \left(\frac{d\mathbf{L}}{dt}\right)_{S_0} = \left(\frac{d\mathbf{L}}{dt}\right)_S + \boldsymbol{\omega} \times \mathbf{L}. \tag{21}$$

Thus, in the body frame the equation of motion becomes

$$\boldsymbol{\tau} = \dot{\mathbf{L}} + \boldsymbol{\omega} \times \mathbf{L}. \tag{22}$$

This equation is called **Euler's equation**.

Euler's equation is much easier to deal with if we choose a body frame whose axes align with the principal axes of the body. In this case the moment of inertia tensor is diagonalized, and the angular momentum is

$$\mathbf{L} = I\boldsymbol{\omega} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \boldsymbol{\omega} = \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix}. \tag{23}$$

Differentiating this and plugging it into eq. (22) gives

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \lambda_1\dot{\omega}_1 \\ \lambda_2\dot{\omega}_2 \\ \lambda_3\dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \times \begin{bmatrix} \lambda_1\omega_1 \\ \lambda_2\omega_2 \\ \lambda_3\omega_3 \end{bmatrix}. \tag{24}$$

Expanding out the cross-product, we get a (nonlinear) system of differential equations

$$\begin{cases} \lambda_1\dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 = \tau_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 = \tau_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 = \tau_3 \end{cases} \tag{25}$$

that describe the time evolution of the components of the angular velocity vector $\boldsymbol{\omega}$.

It is difficult to solve these differential equation in general, but some special cases lead to important insights about rotational motion.

2.5.1 Rotation About a Principal Axis

Suppose a rigid body is initially rotating a about a principal axis. We might as well identify this as axis 1, in which case we have $\boldsymbol{\omega} = (\omega_1, 0, 0)$ and eq. (25) becomes (at least initially)

$$\begin{cases} \lambda_1 \dot{\omega}_1 = \tau_1 \\ \lambda_2 \dot{\omega}_2 = \tau_2 \\ \lambda_3 \dot{\omega}_3 = \tau_3. \end{cases} \tag{26}$$

In the absence of any external torque we have $\tau_1 = \tau_2 = \tau_3 = 0$ and this system implies $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. Thus the angular velocity vector will be constant, and motion of the object is a simple, constant rotation about a fixed axis.

2.5.2 Rotation with Zero Torque

10.8

For free rotation in the absence of an external torque, we have $\tau_1 = \tau_2 = \tau_3 = 0$ and eq. (25) becomes

$$\begin{cases} \dot{\omega}_1 = \frac{\lambda_2 - \lambda_3}{\lambda_1} \omega_2 \omega_3 \\ \dot{\omega}_2 = \frac{\lambda_3 - \lambda_1}{\lambda_2} \omega_3 \omega_1 \\ \dot{\omega}_3 = \frac{\lambda_1 - \lambda_2}{\lambda_3} \omega_1 \omega_2. \end{cases} \tag{27}$$

It isn't hard to show, using these equations, that $\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2 = T$ (the total rotational kinetic energy) is constant. This equation describes the surface of an ellipsoid in $(\omega_1, \omega_2, \omega_3)$-space. As the body rotates, the vector $\boldsymbol{\omega}$ moves around on this ellipsoid.

It is possible to infer the nature of the motion directly from equations (27). I will only describe the key results; Sec. 10.8 gives some of the details. The key ideas are demonstrated nicely in the videos with links below.

In general a rigid body will have three principal axes and corresponding moments of inertia that we can order as $\lambda_1 < \lambda_2 < \lambda_3$. With $\boldsymbol{\omega}$ aligned close to either axis 1 or axis 3, the rotation is stable: the motion of the body will be a rotation in which the axis of rotation $\boldsymbol{\omega}$ precesses slowly about the principal axis. This **free precession** is described mathematically on pp. 398–400, and accounts (among other things) for the *Chandler wobble* of Earth's axis of rotation.

The interesting case is for rotation about axis 2, the so-called intermediate axis. This rotation is unstable: the axis of rotation will spontaneously flip, repeatedly and sometimes rapidly, leading to a wild tumbling motion—a result known as the **intermediate axis theorem**. An excellent demonstration can be seen in this video (recorded in zero gravity):

`https://www.youtube.com/watch?v=GgVp0orcKqc`

This one gives a good explanation:

https://www.youtube.com/watch?v=-Si6iRL5Fj8

and this one is super cool (especially the demo around 5:15) and more detailed:

https://www.youtube.com/watch?v=1VPfZ_XzisU&t=774s

April 2

3 Hamiltonian Mechanics

Ch. 13

3.1 The Hamiltonian Function

Hamiltonian mechanics is an alternative formulation of the Lagrangian mechanics approach we studied earlier. It has many theoretical advantages, and is especially important in drawing the connection to quantum mechanics. We won't go that far into the subject, and indeed there is much more than can be said about e.g. Poisson brackets, generators of group actions, symplectic manifolds, symmetries and Lie algebras, at a more advanced level.

In the Lagrangian description, a mechanical system is described by a set of coordinates $q_1, \ldots, q_n$. At any given instant in time the state of the system is described by the vector $(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ of **coordinates and their velocities**. The resulting equations of motion (which we get via the Euler-Lagrange equations) are a system of 2nd-order differential equations in terms of the $\dot{q}_i$ and $\ddot{q}_i$ $(i = 1, \ldots, n)$.

In the Hamiltonian description, the momenta play a more prominent role; in fact we eliminate the velocities from the description and use the momenta instead. At any given instant in time the state of the system is described by the vector $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ of **coordinates and their momenta**.

Recall that the momentum conjugate to coordinate q_i is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{28}$$

The equations of motion are then derived from the **Hamiltonian function**, which is defined as

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i \dot{q}_i - L \tag{29}$$

where $L = T - V$ is the usual Lagrangian. Although eq. (29) defines H in terms of the $\dot{q}_i$, in practice it is important to use (28) to eliminate the $\dot{q}_i$ in favor of p_i.

Example: Find the Hamiltonian function for the 1-D motion of a mass m attached to a fixed spring of stiffness k.

Solution: We'll take coordinate x to be the displacement of the mass from it's equilibrium position (i.e. from the rest length of the spring). Then the Lagrangian function is

$$L = T - V \tag{30}$$

$$= \tfrac{1}{2} m \dot{x}^2 - \tfrac{1}{2} k x^2. \tag{31}$$

The momentum conjugate to x is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{32}$$

(which in this case is the usual linear momentum). Using eq. (29) we get the Hamiltonian function

$$\begin{aligned} H &= p\dot{x} - L \qquad (33) \\ &= p\dot{x} - \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2. \qquad (34) \end{aligned}$$

However, it is critical that we eliminate $\dot{x}$ from this function so that H is a function of x and p *only*. To do this, use eq. (32) to write

$$\dot{x} = \frac{p}{m} \qquad (35)$$

which can be used to eliminate $\dot{x}$ in eq. (34) to get

$$\begin{aligned} H &= p\left(\frac{p}{m}\right) - \tfrac{1}{2}m\left(\frac{p}{m}\right)^2 + \tfrac{1}{2}kx^2 \qquad (36) \\ &= \boxed{\frac{p^2}{2m} + \tfrac{1}{2}kx^2.} \qquad (37) \end{aligned}$$

Note that in this case $H = T + V$ is just the total energy of the system. In fact this is usually the case (and makes it much easier to find H) for reasons we'll discuss later.

Reading Assignment: Sec. 13.1

3.2 Hamiltonian Equations of Motion

Sec. 13.2

In the Hamiltonian framework, the equations of motion are especially simple:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \qquad (38)$$

Why? We will derive these for a system with one coordinate only; the generalization to more coordinates is fairly obvious. In the example above we eliminated $\dot{x}$ from the Hamiltonian by solving $p = \frac{\partial L}{\partial \dot{x}}$ for $\dot{x}$ to get $\dot{x} = p/m$. In general we would solve $p = \frac{\partial L}{\partial \dot{q}}$ to write $\dot{q}$ as some function $\dot{q}(q,p)$. In this notation, eq. (29) gives the Hamiltonian as

$$\begin{aligned} H &= p\dot{q} - L(q,\dot{q}) \qquad (39) \\ &= p\dot{q}(q,p) - L(q,\dot{q}(q,p)). \qquad (40) \end{aligned}$$

The notation is a bit cumbersome, but gives H as a function of q, p only. Now consider the partial derivative $\frac{\partial H}{\partial q}$. Because p is independent of q we get (using the chain rule):

$$\begin{aligned} \frac{\partial H}{\partial q} &= p\frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial q} \qquad (41) \\ &= -\frac{\partial L}{\partial q} \qquad (42) \\ &= -\frac{d}{dt}\underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} \qquad \text{(by Euler-Lagrange)} \qquad (43) \end{aligned}$$

so that

$$\dot{p} = -\frac{\partial H}{\partial q}. \tag{44}$$

On the other hand, eq. (40) gives the partial derivative $\frac{\partial H}{\partial p}$ as

$$\frac{\partial H}{\partial p} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p} \tag{45}$$

$$= \dot{q}. \tag{46}$$

A key difference between the Lagrangian and Hamiltonian formulations is that the equations of motion are now *first*-order differential equations, but there are twice as many equations. In vector form the equations of motion are

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right). \tag{47}$$

The right-hand side defines a vector field on the **phase space** with coordinates (q, p).

Reading Assignment: Sec. 13.2

Example: Derive the Hamiltonian equations of motion for the mass-spring system of the previous example.

Solution: We already have the Hamiltonian function

$$H(x, p) = \frac{p^2}{2m} + \tfrac{1}{2}kx^2. \tag{48}$$

The equations of motion are then

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} = -kx. \end{cases} \tag{49}$$

We can write these in vector form as

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} p/m \\ -kx \end{bmatrix} = \begin{bmatrix} 0 & 1/m \\ -k & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}. \tag{50}$$

As expected, the equations of motion are a system of first-order autonomous differential equations for the unknown functions $x(t)$, $p(t)$. The solution of the equations (hence the motion of the system) corresponds to a curve (called a **phase space orbit**) in the 2-dimensional **phase space** with coordinates (x, p). In this particular case the differential equations are linear, so could be solved using methods from MATH 2240.

Reading Assignment: Sec. 11.2, 11.3

Exercises: Ch. 13; #2, 3, 4, 5, 6, 9, 12, 25, 26

3.3 Conservation of Energy

In general, depending on the coordinate system used to describe a mechanical system, the Hamiltonian might depend explicitly on time:

$$H = H(q_1, \dots, q_n, p_1, \dots, p_n, t). \tag{51}$$

Consider the time derivative $\frac{dH}{dt}$ (via the chain rule):

$$\begin{aligned}
\frac{dH}{dt} &= \underbrace{\frac{\partial H}{\partial q_1}}_{-\dot{p}_1} \frac{dq_1}{dt} + \underbrace{\frac{\partial H}{\partial q_2}}_{-\dot{p}_2} \frac{dq_2}{dt} + \cdots + \underbrace{\frac{\partial H}{\partial p_1}}_{\dot{q}_1} \frac{dp_1}{dt} + \underbrace{\frac{\partial H}{\partial p_2}}_{\dot{q}_2} \frac{dp_2}{dt} + \cdots + \frac{\partial H}{\partial t} \\
&= (-\dot{p}_1 \dot{q}_1 + \dot{q}_1 \dot{p}_1) + (-\dot{p}_2 \dot{q}_2 + \dot{q}_2 \dot{p}_2) + \cdots + \frac{\partial H}{\partial t} \\
&= \frac{\partial H}{\partial t}.
\end{aligned}$$

Consequently, if H does not depend *explicitly* on time ($\frac{\partial H}{\partial t} = 0$) then the time rate of change of H is zero. In other words, the quantity H is constant throughout the motion.

3.4 Natural Coordinates

It is always possible to write the coordinates $q_1, \dots, q_n$ in terms of the usual *Cartesian* coordinates x, y, z. If this relationship to the underlying Cartesian coordinates is independent of time, then the coordinates are said to be **natural**. It turns out that, provided the coordinates $q_1, \dots, q_n$ are natural, then the Hamiltonian is equal to the total energy:

$$H = T + V. \tag{52}$$

(The derivation is a bit lengthy; see Sec. 7.8 for details).

Combining this with the result of the previous section: if the Hamiltonian expressed in terms of natural coordinates is not explicitly time-dependent, then the total energy in the system ($H = T + V$) is conserved, i.e. it is a constant of motion.

The End.