Thompson Rivers University

MATH 316
Differential Equations II

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MIDTERM EXAM #1
SOLUTIONS

19 October 2011 10:30–12:20

PROBLEM	GRADE	OUT OF
1		10
2		5
3		4
4		10
5		5
6		8
7		8
TOTAL:		50

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 7 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

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Problem 1: Consider the differential equation $y'' + xy' + 2y = 0$.

/2 (a) Classify $x = 0$ as either an ordinary point or a singular point. Explain.

$$y'' + \underbrace{x}_{P(x)} y' + \underbrace{2}_{Q(x)} y = 0$$

$P(x) = x$ and $Q(x) = 2$ are both analytic functions (at $x = 0$ in particular) so $x = 0$ is an ordinary point.

/8 (b) Find two linearly independent solutions, expressed as power series about $x = 0$. Simplify as much as possible.

Assume $y = \sum_{n=0}^{\infty} c_n x^n \quad \Longrightarrow \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \Longrightarrow \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

Sub into the DE:

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

Pull out $n = 0$ terms:

$$2c_2 + 2c_0 + \sum_{n=1}^{\infty} \big[(n+2)(n+1)c_{n+2} + n c_n + 2c_n\big] x^n = 0$$

This gives $c_2 = -c_0$ together with the recurrence relation: $c_{n+2} = -\dfrac{c_n}{n+1}$.

$$c_4 = -\frac{c_2}{3} = \frac{c_0}{1\cdot 3} \qquad c_3 = -\frac{c_1}{2}$$

$$c_6 = -\frac{c_4}{5} = -\frac{c_0}{1\cdot 3\cdot 5} \qquad c_5 = -\frac{c_3}{4} = \frac{c_1}{2\cdot 4}$$

$$\vdots \qquad \vdots$$

$$c_{2m} = \frac{(-1)^m c_0}{1\cdot 3\cdot 5\cdots(2m-1)} \qquad c_{2m+1} = \frac{(-1)^m c_1}{2\cdot 4\cdot 6\cdots(2m)} = \frac{(-1)^m c_1}{2^m m!}$$

$$\Longrightarrow y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{1\cdot 3\cdot 5\cdots(2m-1)} x^{2m}}_{y_1(x)} + c_1 \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{2^m m!} x^{2m+1}}_{y_2(x)}$$

This gives an arbitrary linear combination of y_1, y_2 which are two linearly independent solutions.

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Problem 2: Use the Laplace Transform to solve the initial value problem

$$y' = \delta(t-a), \qquad y(0) = 0.$$

Explain how this demonstrates that $\frac{d}{dt}u(t-a) = \delta(t-a)$.

$$\xrightarrow{\mathcal{L}} \quad sY = e^{-as} \implies Y(s) = \frac{1}{s}e^{-as}$$

$$\xrightarrow{\mathcal{L}^{-1}} \quad y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-as}\right\} = u(t-a)$$

Since $y(t) = u(t-a)$ is a solution of $\frac{dy}{dt} = \delta(t-a)$ we have that $\frac{d}{dt}u(t-a) = \delta(t-a)$.

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Problem 3: Find the radius of convergence and interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$.

To apply the ratio test we let $a_n = \frac{(-2)^n x^n}{\sqrt[4]{n}}$ and calculate

$$\begin{aligned} L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(-2)^{n+1}x^{n+1}/(n+1)^{1/4}}{(-2)^n x^n/n^{1/4}} \\ &= 2|x|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{1/4} \\ &= 2|x| \end{aligned}$$

The series converges if

$$L = 2|x| < 1 \implies |x| < \frac{1}{2}$$

Thus the radius of convergence is $R = \frac{1}{2}$ and the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

(Note: the interval of convergence may actually contain one or both endpoints; we'll simply neglect that issue here. What you would need to do is form the series with $x = \pm\frac{1}{2}$ and apply some appropriate test for convergence.)

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Problem 4: Consider the differential equation

$$3xy'' + 2y' + 2y = 0.$$

(a) Show that $x = 0$ is a regular singular point for this equation.

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$$y'' + \underbrace{\frac{2}{3x}}_{P(x)} y' + \underbrace{\frac{2}{3x}}_{Q(x)} y = 0$$

$x = 0$ is a singular point because both $P(x)$ and $Q(x)$ singular (hence not analytic) there, but both $xP(x) = \frac{2}{3}$ and $x^2Q(x) = \frac{2}{3}x$ are analytic (at $x = 0$ in particular) so $x = 0$ is a *regular* singular point.

(b) Find the general solution, expressed as a series about $x = 0$.

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Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r} \implies y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \implies y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$

Sub into the DE:

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1} + \underbrace{\sum_{n=0}^{\infty} 2c_n x^{n+r}}_{\sum_{n=1}^{\infty} 2c_{n-1}x^{n+r-1}} = 0$$

Pull out the $n = 0$ terms:

$$\Big[3r(r-1) + 2r\Big] c_0 x^{r-1} + \sum_{n=1}^{\infty} \Big[3(n+r)(n+r-1)c_n + 2(n+r)c_n + 2c_{n-1}\Big] x^{n+r-1}$$

This gives the indicial equation $0 = 3r(r-1) + 2r = r(3r-1) \implies r = 0, \frac{1}{3}$

and the recurrence relation $c_n = \dfrac{-2c_{n-1}}{(n+r)(3n+3r-1)}$.

Case $r = \frac{1}{3}$: $\quad c_n = \dfrac{-2c_{n-1}}{(n+\frac{1}{3})(3n)} = \dfrac{-2c_{n-1}}{(3n+1)n}$

$$c_1 = \frac{-2c_0}{4 \cdot 1} \implies c_2 = \frac{-2c_1}{7 \cdot 2} = \frac{(-2)^2 c_0}{(4 \cdot 7)(1 \cdot 2)} \implies c_3 = \frac{-2c_2}{10 \cdot 3} = \frac{(-2)^3 c_0}{(4 \cdot 7 \cdot 10)(1 \cdot 2 \cdot 3)} \implies \cdots$$

$$c_n = \frac{(-2)^n c_0}{n! \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)}$$

Case $r = 0$: $\quad c_n = \dfrac{-2c_{n-1}}{n(3n-1)}$

$$c_1 = \frac{-2c_0}{1 \cdot 2} \implies c_2 = \frac{-2c_1}{2 \cdot 5} = \frac{(-2)^2 c_0}{(1 \cdot 2)(2 \cdot 5)} \implies c_3 = \frac{-2c_2}{3 \cdot 8} = \frac{(-2)^3 c_0}{(1 \cdot 2 \cdot 3)(2 \cdot 5 \cdot 8)} \implies \cdots$$

$$c_n = \frac{(-2)^n c_0}{n! \cdot 2 \cdot 5 \cdot 8 \cdots (3n-1)}$$

$$\implies \boxed{y(x) = A\sum_{n=0}^{\infty} \frac{(-2)^n}{n! \cdot 4 \cdot 7 \cdot 10 \cdots (3n+1)} x^{n+\frac{1}{3}} + B\sum_{n=0}^{\infty} \frac{(-2)^n}{n! \cdot 2 \cdot 5 \cdot 8 \cdots (3n-1)} x^n}$$

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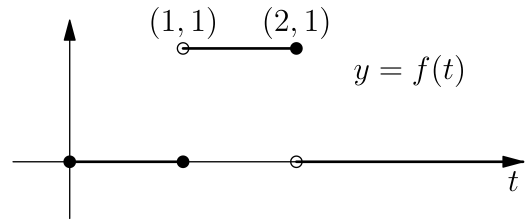
Problem 5: Consider the function $f(t)$ whose graph is shown.



(a) Use the *definition* of the Laplace transform to find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ by evaluating an appropriate integral.

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$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_1^2 e^{-st}(1)\, dt = -\frac{1}{s}e^{-st}\Big|_{t=1}^{t=2} = \boxed{\frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}}$$

(b) Express $f(t)$ in terms of Heaviside step functions. Show how to find $F(s)$ without evaluating an integral, by using $\mathcal{L}\{u(t-a)\} = \dfrac{e^{-as}}{s}$.

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$$f(t) = u(t-1) - u(t-2)$$

$$\implies F(s) = \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s}$$

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Problem 6: Use the Laplace Transform to solve the following initial value problem:

$$y'' + 4y = f(t) \qquad \text{where} \qquad f(t) = \begin{cases} \cos(t) & \text{if } 0 \le t < 2\pi \\ 0 & \text{if } t \ge 2\pi. \end{cases}$$
$$y(0) = y'(0) = 0$$

$$\begin{aligned} f(t) &= \cos t - u(t-2\pi)\cos t \\ &= \cos t - u(t-2\pi)\cos(t-2\pi) \quad \text{(by periodicity of cos)} \end{aligned}$$

$$\Longrightarrow\ F(s) = \frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1}$$

Laplace transform the DE:

$$s^2Y + 4Y = \frac{s}{s^2+1} - e^{-2\pi s}\frac{s}{s^2+1} \Longrightarrow Y(s) = \underbrace{\frac{s}{(s^2+4)(s^2+1)}}_{G(s)} - e^{-2\pi s}\underbrace{\frac{s}{(s^2+4)(s^2+1)}}_{G(s)}$$

$$\Longrightarrow\ y(t) = g(t) - u(t-2\pi)g(t-2\pi).$$

Partial fractions:

$$G(s) = \frac{s}{(s^2+4)(s^2+1)} = \frac{As+b}{s^2+4} + \frac{Cs+D}{s^2+1} = \frac{(As+b)(s^2+1)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

and matching coefficients in the numerators gives:

$$\begin{aligned} s^3:&\quad A + C = 0 \Longrightarrow C = -A\ (= 1/3) \\ s^2:&\quad B + D = 0 \Longrightarrow D = -B\ (= 0) \\ s:&\quad A + 4C = 1 \Longrightarrow -3A = 1 \Longrightarrow A = -1/3 \\ 1:&\quad B + 4D = 0 \Longrightarrow -3D = 0 \Longrightarrow D = 0 \end{aligned}$$

so

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\{G(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}\right\} \\ &= \frac{1}{3}\cos t - \frac{1}{3}\cos(2t) \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ y(t) &= \frac{1}{3}\Big(\cos t - \cos 2t - u(t-2\pi)\big[\cos(t-2\pi) - \cos(2(t-2\pi))\big]\Big) \\ &= \frac{1}{3}\Big(\cos t - \cos 2t - u(t-2\pi)\big[\cos t - \cos 2t\big]\Big) \quad \text{(by periodicity of cos)} \end{aligned}$$

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Problem 7: Use the Laplace Transform to solve the following initial value problem:

$$y'' + 2y' + y = \delta(t-1) - \delta(t-2), \qquad y(0) = y'(0) = 2.$$

Laplace transform the DE:

$$(s^2 Y - 2s - 2) + 2(sY - 2) + Y = e^{-s} - e^{-2s}$$

$$\Longrightarrow (s^2 + 2s + 1)Y = 2s + 6 + e^{-s} - e^{-2s}$$

$$\Longrightarrow Y(s) = \frac{2s+6}{(s+1)^2} + e^{-s}\frac{1}{(s+1)^2} - e^{-2s}\frac{1}{(s+1)^2}$$

Invert the transform:

$$\Longrightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{2s+6}{(s+1)^2}\right\} + \mathcal{L}^{-1}\Bigg\{e^{-s}\underbrace{\frac{1}{(s+1)^2}}_{G(s)}\Bigg\} - \mathcal{L}^{-1}\Bigg\{e^{-2s}\underbrace{\frac{1}{(s+1)^2}}_{G(s)}\Bigg\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2(s+1)+4}{(s+1)^2}\right\} + u(t-1)g(t-1) - u(t-2)g(t-2)$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{2s+4}{s^2}\right\} + u(t-1)g(t-1) - u(t-2)g(t-2)$$

$$= e^{-t}(2+4t) + u(t-1)g(t-1) - u(t-2)g(t-2)$$

$$\text{where} \quad g(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}$$

$$\Longrightarrow \boxed{y(t) = (4t+2)e^{-t} + u(t-1)(t-1)e^{-(t-1)} - u(t-2)(t-2)e^{-(t-2)}}$$