THOMPSON RIVERS UNIVERSITY

MATH 3160
Differential Equations II

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FINAL EXAM
SOLUTIONS

5 December 2011 14:00–17:00

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 10 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		10
3		7
4		8
5		8
6		11
7		10
8		10
9		11
TOTAL:		85

/10 **Problem 1:** Consider the differential equation

$$(x^2-4)y''+3xy'+y=0.$$

(a) Classify the point $x=0$ as an ordinary point, regular singular point, or irregular singular point. Justify your answer carefully.

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$$y''+\frac{3x}{x^2-4}y'+\frac{1}{x^2-4}y=0$$

Both $P(x)$, $Q(x)$ are analytic at $x=0$ (rational functions away from their poles) so $x=0$ is an ordinary point.

(b) Find two linearly independent solutions expressed as series about $x=0$.

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Assume $y=\sum_{n=0}^{\infty}c_nx^n$, sub. into the the DE and re-index as needed:

$$\implies \sum_{n=2}^{\infty}n(n-1)c_nx^n-\underbrace{\sum_{n=2}^{\infty}4n(n-1)x^{n-2}}_{\sum_{n=0}^{\infty}4(n+2)(n+1)c_{n+2}x^n}+\sum_{n=1}^{\infty}3nc_nx_n+\sum_{n=0}^{\infty}c_nx^n=0$$

Collect terms:

$$-8c_2-24c_3x+3c_1x+c_0+c_1x+\sum_{n=2}^{\infty}\left[n(n-1)c_n-4(n+2)(n+1)c_{n+2}+3nc_n+c_n\right]x^n=0$$

Zero'th and first-order terms give

$$-8c_2+c_0=0\implies c_2=\frac{1}{8}c_0$$
$$-24c_3+4c_1=0\implies c_3=\frac{1}{6}c_1$$

while the other terms give the recursion relation:

$$c_{n+2}=\frac{n(n-1)+3n+1}{4(n+2)(n+1)}c_n=\frac{n+1}{4(n+2)}c_n$$

Thus

$$c_4=\frac{3}{4\cdot 4}c_2=\frac{3}{8\cdot 4\cdot 4}c_0 \qquad c_5=\frac{4}{4\cdot 5}c_3=\frac{4}{6\cdot 4\cdot 5}c_1$$
$$c_6=\frac{5}{4\cdot 6}c_4=\frac{3\cdot 5}{8\cdot 4^2\cdot 4\cdot 6}c_0=\frac{3\cdot 5}{4^3\cdot 2\cdot 4\cdot 6}c_0 \qquad c_7=\frac{6}{4\cdot 7}c_5=\frac{4\cdot 6}{6\cdot 4^2\cdot 5\cdot 7}c_1=\frac{2\cdot 4\cdot 6}{4^3\cdot 3\cdot 5\cdot 7}c_1$$
$$c_8=\frac{7}{4\cdot 8}c_6=\frac{3\cdot 5\cdot 7}{4^4\cdot 2\cdot 4\cdot 6\cdot 8}c_0 \qquad c_9=\frac{8}{4\cdot 9}c_7=\frac{2\cdot 4\cdot 6\cdot 8}{4^4\cdot 3\cdot 5\cdot 7\cdot 9}c_1$$
$$\vdots \qquad \vdots$$
$$c_{2m}=\frac{3\cdot 5\cdot 7\cdots(2m-1)}{4^m\cdot 2\cdot 4\cdot 6\cdots(2m)}c_0=\frac{(2m)!}{4^{2m}(m!)^2}c_0 \qquad c_{2m+1}=\frac{2\cdot 4\cdot 6\cdots(2m)}{4^m\cdot 3\cdot 5\cdot 7\cdots(2m+1)}c_1=\frac{(m!)^2}{(2m+1)!}c_1$$

So

$$y(x)=c_0\underbrace{\left[1+\sum_{m=1}^{\infty}\frac{(2m)!}{4^{2m}(m!)^2}x^{2m}\right]}_{y_0(x)}+c_1\underbrace{\left[x+\sum_{m=1}^{\infty}\frac{(m!)^2}{(2m+1)!}x^{2m+1}\right]}_{y_1(x)}$$

/10 **Problem 2:** In many applications it is useful to represent a function $y(x)$ as a series about the "point at infinity" (e.g. if one needs to approximate the solution $y(x)$ for large values of x). Consider the differential equation

$$x^3y'' - x^2y' - y = 0.$$

(a) Show that with $z = 1/x$ this equation can be written $z\dfrac{d^2y}{dz^2} + 3\dfrac{dy}{dz} - y = 0$. (Note that $z = 0$ is a regular singular point; we say the original equation has a regular singular point "at infinity".)

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The chain rule gives

$$\frac{d}{dx} = \frac{dz}{dx}\frac{d}{dz} = -x^{-2}\frac{d}{dz} = -z^2\frac{d}{dz}$$

Thus the DE becomes

$$x^3(-z^2)\frac{d}{dz}\Big(-z^2\frac{dy}{dz}\Big) - x^2(-z^2)\frac{dy}{dz} - y = 0$$

$$\implies \underbrace{x^3z^4}_{z}\frac{d^2y}{dz^2} + \underbrace{x^3(-z^2)(-2z)}_{2}\frac{dy}{dz} + \underbrace{x^2z^2}_{1}\frac{dy}{dz} - y = 0$$

$$\implies z\frac{d^2y}{dz^2} + 3\frac{dy}{dz} - y = 0$$

(b) Find the first three non-zero terms in a series solution about $z = 0$.

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Method of Frobenius: assume $y = \sum_{n=0}^{\infty} c_n z^{n+r}$. Sub. into the DE and re-index:

$$\implies \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)z^{n+r-1} + \sum_{n=0}^{\infty} 3c_n(n+r)z^{n+r-1} - \underbrace{\sum_{n=0}^{\infty} c_n z^{n+r}}_{\sum_{n=1}^{\infty} c_{n-1}z^{n+r-1}} = 0$$

$$\implies \big[r(r-1)+3r\big]c_0z^{r-1} + \sum_{n=1}^{\infty}\big[c_n(n+r)(n+r-1) + 3c_n(n+r) - c_{n-1}\big]z^{n+r-1} = 0$$

This gives the indicial equation:

$$0 = r(r-1) + 3r = r^2 + 2r \implies r = 0, -2$$

and the recurrence relation (with $r = 0$):

$$c_n = \frac{c_{n-1}}{n(n-1)+3n} = \frac{1}{n(n+2)}c_{n-1}$$

Iterating the recurrence relation gives

$$c_1 = \frac{1}{3}c_0, \qquad c_2 = \frac{1}{2\cdot 4}c_1 = \frac{1}{24}c_0$$

so

$$y = c_0\left(1 + \frac{1}{3}z + \frac{1}{24}z^2 + \cdots\right)$$

(c) Rewrite your answer to part (b) with x as the independent variable. Explain why your answer approximates $y(x)$ for x "near infinity".

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$$y = c_0\left(1 + \frac{1}{3x} + \frac{1}{24x^2} + \cdots\right)$$

For sufficiently large x (i.e. x "near infinity") the higher-order terms become insignificant, so the truncated series is a good approximation of the solution.

/7 **Problem 3:** The gamma function is defined as $\Gamma(t) = \int_0^\infty e^{-u} u^{t-1}\, du \quad (t \in \mathbb{R},\ t > 0)$.

(a) Prove the identity $\Gamma(t+1) = t\Gamma(t)$. (Hint: integrate by parts.)

(You might find it interesting that this implies $\Gamma(n+1) = n!$ for non-negative integers n, hence Γ generalizes the factorial function to non-integer values of its argument.)

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From the definition,

$$\begin{aligned}\Gamma(t+1) &= \int_0^\infty e^{-u} u^t\, du \qquad \text{(int. by parts with } v = u^t,\ dw = e^{-u}\, du) \\ &= \underbrace{\lim_{b\to\infty} -e^{-u} u^t \Big|_{u=0}^{b}}_{0} + \int_0^\infty e^{-u} t u^{t-1}\, du \\ &= t \int_0^\infty e^{-u} u^{t-1}\, du \equiv t\Gamma(t)\end{aligned}$$

(b) Use the definition of the Laplace transform to show that for any real $r > -1$, the function $f(t) = t^r$ has Laplace transform

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$$\mathcal{L}\{t^r\} = \frac{\Gamma(r+1)}{s^{r+1}}$$

$$\begin{aligned}\mathcal{L}\{t^r\} &= \int_0^\infty e^{-st} t^r\, dt \qquad \text{(substitute } u = st) \\ &= \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^r \frac{du}{s} \\ &= \frac{1}{s^{r+1}} \underbrace{\int_0^\infty e^{-u} u^r\, du}_{\Gamma(r+1)} \\ &= \frac{\Gamma(r+1)}{s^{r+1}}\end{aligned}$$

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Problem 4: Solve the following initial value problem:

$$y'' - y = 4\delta(t-2) + t^2$$
$$y(0) = 0, \quad y'(0) = 3$$

Laplace transform:

$$(s^2Y - s\cdot 0 - 3) - Y = 4e^{-2s} + \frac{2}{s^3} \implies (s^2-1)Y = 3 + 4e^{-2s} + \frac{2}{s^3}$$

$$\implies Y(s) = 3\underbrace{\frac{1}{s^2-1}}_{W(s)} + 4e^{-2s}\underbrace{\frac{1}{s^2-1}}_{W(s)} + \underbrace{\frac{2}{s^3(s^2-1)}}_{G(s)}$$

So

$$y(t) = 3w(t) + 4u(t-2)w(t-2) + g(t)$$

where

$$w(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right\}$$
$$= \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh(t)$$

and

$$g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^3(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{2}{s^3} - \frac{2}{s} + \frac{1}{s-1} + \frac{1}{s+1}\right\}$$
$$= -t^2 - 2 + e^t + e^{-t}$$
$$= -t^2 - 2 + 2\cosh(t)$$

so

$$y(t) = 3\sinh(t) + 4u(t-2)\sinh(t-2) + 2\cosh(t) - t^2 - 2$$

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Problem 5: Solve the following initial value problem, expressing your answer as an integral involving $f(t)$:

$$y'' + 4y' + 8y = f(t)$$
$$y(0) = 1, \quad y'(0) = 0$$

Laplace transform:

$$(s^2Y - s \cdot 1 - 0) + 4(sY - 1) + 8Y = F(s) \implies (s^2 + 4s + 8)Y = s + 4 + F(s)$$

$$\implies Y(s) = \underbrace{\frac{s+4}{s^2+4s+8}}_{W(s)} + \underbrace{\frac{1}{s^2+4s+8}}_{G(s)} F(s)$$

So

$$y(t) = w(t) + g * f(t)$$

where

$$w(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2^2} + \frac{2}{(s+2)^2+2^2}\right\}$$
$$= e^{-2t}\cos 2t + e^{-2t}\sin 2t$$

and

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{2}{(s+2)^2+2^2}\right\}$$
$$= \frac{1}{2}e^{-2t}\sin 2t$$

so

$$y(t) = e^{-2t}\cos 2t + e^{-2t}\sin 2t + \frac{1}{2}\int_0^t e^{-2\tau}\sin 2\tau\, f(t-\tau)\, d\tau$$

/11 **Problem 6:** The "advection equation"

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(c(x) \cdot u) = 0 \quad (-\infty < x < \infty,\ t > 0)$$

arises in the study of fluid dynamics: it describes the concentration $u(x,t)$ of a substance carried by a moving fluid having velocity $c(x)$ along a one-dimensional pipe.

(a) Use separation of variables to find the general solution of the advection equation when $c(x) = c$ is a constant.

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(b) Use separation of variables to find the general solution of the advection equation with $c(x) = x$.

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(c) Show that for any function f that is differentiable, $u(x,t) = f(x - ct)$ is a solution of the advection equation when $c(x) = c$ is a constant. (This is called "d'Alembert's solution").

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Problem 7: Solve the following initial boundary value problem for $u(x,t)$. (Note the non-homogeneous boundary conditions.)

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} \qquad (0 < x < L,\ t > 0)\\
&u(0,t) = 0, \quad u(L,t) = P \text{ (a constant)}\\
&u(x,0) = f(x)\\
&u_t(x,t) = g(x)
\end{aligned}
$$

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Problem 8: Solve the following initial boundary value problem for $u(x,t)$:

$$u_t = 7u_{xx}, \quad 0 < x < \frac{\pi}{2},\ t > 0$$
$$u_x(0,t) = 0, \quad u(\pi/2,t) = 0$$
$$u(x,0) = 1$$

/11 **Problem 9:** Consider the Sturm-Liouville problem

$$y'' + 2y' + y = -\lambda y, \qquad y(0) = y(1) = 0.$$

/2 (a) Write the differential equation in self-adjoint form.

/5 (b) Determine the eigenvalues λ and corresponding eigenfunctions.

/2 (c) Give an orthogonality relation for the eigenfunctions for this problem.

/2 (d) Express the function e^x $(0 < x < 1)$ as a linear combination of the eigenfunctions you found. (Simplify but do not evaluate the definite integral(s) that appear your answer).