MATH 3160: Quiz #3 – SOLUTIONS

/5 **Problem 1:** Use the definition of the Laplace Transform $\mathcal{L}$ to find $F(s) = \mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} t & 0 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases}$

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\ &= \int_0^2 e^{-st} t\,dt \qquad \text{int. by parts: } \left\{ \begin{array}{ll} u = t & dv = e^{-st}\,dt \\ du = dt & v = -\frac{1}{s}e^{-st} \end{array} \right. \\ &= \left[-\frac{t}{s}e^{-st}\right]_{t=0}^{2} + \int_0^2 \frac{1}{s}e^{-st}\,dt \\ &= -\frac{2}{s}e^{-2s} + \left[-\frac{1}{s^2}e^{-st}\right]_{t=0}^{2} = \boxed{-\frac{2e^{-2s}}{s} + \frac{1}{s^2} - \frac{e^{-2s}}{s^2}} \end{aligned}$$

check:
$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}e^{-2s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}e^{-2s}\right\} \\ &= t - 2u(t-2) - (t-2)u(t-2) \\ &= t - tu(t-2) \\ &= t[1 - u(t-2)] \\ &= \begin{cases} t & 0 \le t \le 2 \\ 0 & \text{otherwise.} \end{cases} \qquad \checkmark \end{aligned}$$

/5 **Problem 2:** Use the Laplace transform to find the solution $y(t)$ of the following initial value problem:

$$y'' + 7y' + 10y = 0, \quad y(0) = 0, \quad y'(0) = 2$$

Laplace transform of this IVP gives:

$$[s^2Y - 0 \cdot s - 2] + 7[sY - 0] + 10Y = 0$$

$$\implies Y(s) = \frac{2}{s^2 + 7s + 10} = \frac{2}{(s+5)(s+2)} = \frac{2/3}{s+2} - \frac{2/3}{s+5}$$

$$\begin{aligned} \implies y(t) &= \mathcal{L}^{-1}\left\{\frac{2/3}{s+2} - \frac{2/3}{s+5}\right\} \\ &= \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \\ &= \boxed{\frac{2}{3}e^{-2t} - \frac{2}{3}e^{-5t}} \end{aligned}$$