MATH 3160: Quiz #2 – SOLUTIONS

/10 **Problem 1:** Consider the differential equation $2x^2y'' - xy' + (x^2+1)y = 0$.

(a) Show that $x = 0$ is a regular singular point of this equation.

(b) Use the method of Frobenius to find two linearly independent series solutions about $x = 0$.

(a) Divide through by $2x^2$ to obtain

$$y'' \underbrace{-\frac{1}{2x}}_{P(x)} y' + \underbrace{\frac{x^2+1}{2x^2}}_{Q(x)} y = 0.$$

Clearly $x = 0$ is a singular point since both $P(x)$ and $Q(x)$ fail to be analytic at $x = 0$. But both $xP(x) = -\frac{1}{2}$ and $x^2Q(x) = \frac{1}{2}(x^2+1)$ are analytic at $x = 0$ (since both are polynomials) so $x = 0$ is a *regular* singular point.

(b) Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$ (where without loss of generality we can assume $c_0 \neq 0$).

Then $y' = \sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-2}$. Sub these into the DE:

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Re-index:

$$\sum_{n=0}^{\infty} 2c_n(n+r)(n+r-1)x^{n+r} - \sum_{n=0}^{\infty} c_n(n+r)x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Pull out the $n = 0$ and $n = 1$ terms:

$$\underbrace{[2r(r-1) - r + 1]c_0x^r}_{n=0} + \underbrace{[2(1+r)r - (1+r) + 1]c_1x^{r+1}}_{n=1}$$
$$+ \sum_{n=2}^{\infty} \left[2c_n(n+r)(n+r-1) - c_n(n+r) + c_{n-2} + c_n\right] x^{n+r} = 0. \quad (\star)$$

Since $c_0 \neq 0$ by assumption, the $n = 0$ term gives the "indicial equation"

$$0 = 2r(r-1) - r + 1 = 2r^2 - 3r + 1 \implies \boxed{r = \tfrac{1}{2} \text{ or } r = 1.}$$

<u>**Case $r = 1$:**</u> In this case equation $(\star)$ becomes

$$3c_1x^2 + \sum_{n=2}^{\infty} \left[(2n+1)nc_n + c_{n-2}\right] x^{n+1} = 0$$

which gives the recursion relations

$$\boxed{c_1 = 0, \quad c_n = -\frac{c_{n-2}}{(2n+1)n} \quad (n = 2, 3, \ldots).}$$

$$\begin{aligned}
&\implies && c_1 = c_3 = c_5 = \cdots = 0 \\
&n = 2: && c_2 = -\frac{c_0}{5 \cdot 2} \\
&n = 4: && c_4 = -\frac{c_2}{9 \cdot 4} = \frac{c_0}{(9 \cdot 5)(4 \cdot 2)} \\
&n = 6: && c_6 = -\frac{c_4}{13 \cdot 6} = -\frac{c_0}{(13 \cdot 9 \cdot 5)(6 \cdot 4 \cdot 2)}
\end{aligned}$$

and we get the solution

$$
\begin{aligned}
y(x) &= c_0x^1 + c_1x^2 + c_2x^3 + \cdots \\
&= c_0x^1 - \frac{c_0}{5\cdot 2}x^3 + \frac{c_0}{(9\cdot 5)(4\cdot 2)}x^4 - \frac{c_0}{(13\cdot 9\cdot 5)(6\cdot 4\cdot 2)}x^6 + \cdots \\
&= c_0 \underbrace{\left[x - \frac{1}{5\cdot 2}x^3 + \frac{1}{(9\cdot 5)(4\cdot 2)}x^4 - \frac{1}{(13\cdot 9\cdot 5)(6\cdot 4\cdot 2)}x^6 + \cdots\right]}_{y_0(x)}
\end{aligned}
$$

Case $r = \frac{1}{2}$: In this case equation $(\star)$ becomes

$$c_1x^{3/2} + \sum_{n=2}^{\infty}\left[(2n-1)nc_n + c_{n-2}\right]x^{n+1/2} = 0$$

which gives the recursion relations

$$\boxed{c_1 = 0, \quad c_n = -\frac{c_{n-2}}{(2n-1)n} \quad (n = 2, 3, \ldots).}$$

$$
\begin{aligned}
&\implies & c_1 &= c_3 = c_5 = \cdots = 0 \\
n = 2:& & c_2 &= -\frac{c_0}{3\cdot 2} \\
n = 4:& & c_4 &= -\frac{c_2}{7\cdot 4} = \frac{c_0}{(7\cdot 3)(4\cdot 2)} \\
n = 6:& & c_6 &= -\frac{c_4}{11\cdot 6} = -\frac{c_0}{(11\cdot 7\cdot 3)(6\cdot 4\cdot 2)}
\end{aligned}
$$

and we get the solution

$$
\begin{aligned}
y(x) &= c_0x^{1/2} + c_1x^{3/2} + c_2x^{5/2} + \cdots \\
&= c_0x^{1/2} - \frac{c_0}{3\cdot 2}x^{5/2} + \frac{c_0}{(7\cdot 3)(4\cdot 2)}x^{7/2} - \frac{c_0}{(11\cdot 7\cdot 3)(6\cdot 4\cdot 2)}x^{9/2} + \cdots \\
&= c_0 \underbrace{x^{1/2}\left[1 - \frac{1}{3\cdot 2}x^2 + \frac{1}{(7\cdot 3)(4\cdot 2)}x^4 - \frac{1}{(11\cdot 7\cdot 3)(6\cdot 4\cdot 2)}x^6 + \cdots\right]}_{y_1(x)}
\end{aligned}
$$

Thus the two linearly independent solutions are the functions

$$
\boxed{
\begin{aligned}
y_0(x) &= x - \frac{1}{5\cdot 2}x^3 + \frac{1}{(9\cdot 5)(4\cdot 2)}x^5 - \frac{1}{(13\cdot 9\cdot 5)(6\cdot 4\cdot 2)}x^7 + \cdots \\
y_1(x) &= x^{1/2}\left[1 - \frac{1}{3\cdot 2}x^2 + \frac{1}{(7\cdot 3)(4\cdot 2)}x^4 - \frac{1}{(11\cdot 7\cdot 3)(6\cdot 4\cdot 2)}x^6 + \cdots\right]
\end{aligned}
}
$$

With just a little work we can write these in closed form as

$$
\begin{aligned}
y_0(x) &= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{1\cdot 5\cdot 9\cdots(4m+1)m!2^m} \\
y_1(x) &= x^{1/2} + \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m+\frac{1}{2}}}{3\cdot 7\cdot 11\cdots(4m-1)m!2^m}.
\end{aligned}
$$

Note that $m!2^m$ can also be written as $m!!$.