THOMPSON RIVERS UNIVERSITY

MATH 3160
Differential Equations 2

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MIDTERM EXAM #1
SOLUTIONS

23 Oct 2019 9:30–11:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		9
2		8
3		10
4		5
5		5
TOTAL:		37

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Problem 1: Consider the differential equation $y'' + x^2 y = 0$.

(a) Show that $x = 0$ is an ordinary point of this equation.

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$y'' + P(x)y' + Q(x)y = 0$; coefficients $P(x) = 0$ and $Q(x) = x^2$ are both analytic functions (polynomials) so *every* point is an ordinary point. So, in particular, $x = 0$ is an ordinary point.

(b) Derive a set of recurrence relations for the coefficients of the power series $y = \sum_{n=0}^{\infty} a_n x^n$ such that y is a solution of the given equation.

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$$y = \sum_{n=0}^{\infty} a_n x^n, \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

$$\text{(sub into DE)} \implies \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\text{(re-index)} \implies \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=4}^{\infty} a_{n-4} x^{n-2} = 0$$

$$\text{(pull out leading terms)} \implies 2a_2 + 6a_3 x + \sum_{n=4}^{\infty} \big[n(n-1)a_n + a_{n-4}\big] x^{n-2} = 0$$

$$\text{(equate all coeffs. to 0)} \implies \boxed{\begin{cases} a_2 = 0 \\ a_3 = 0 \\ a_n = -\dfrac{a_{n-4}}{n(n-1)} \quad (n = 4, 5, \ldots) \end{cases}}$$

(c) Solve your recurrence relation to find two linearly independent solutions of this differential equation.

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$$a_2 = a_3 = a_6 = a_7 = \cdots = c_{2m+2} = c_{2m+3} = 0 \quad (m = 0, 1, 2, \ldots)$$

$$a_4 = -\frac{a_0}{4 \cdot 3}, \quad a_8 = -\frac{a_4}{8 \cdot 7} = \frac{a_0}{8 \cdot 7 \cdot 4 \cdot 3}, \quad a_{12} = -\frac{a_8}{12 \cdot 11} = -\frac{a_0}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3}$$

$$a_5 = -\frac{a_1}{5 \cdot 4}, \quad a_9 = -\frac{a_5}{9 \cdot 8} = \frac{a_1}{9 \cdot 8 \cdot 5 \cdot 4}, \quad a_{13} = -\frac{a_9}{13 \cdot 12} = -\frac{a_1}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4}$$

$$y(x) = a_0 \underbrace{\left[1 - \frac{x^4}{4 \cdot 3} + \frac{x^8}{8 \cdot 7 \cdot 4 \cdot 3} - \frac{x^{12}}{12 \cdot 11 \cdot 8 \cdot 7 \cdot 4 \cdot 3} + \cdots\right]}_{y_0(x)} + a_1 \underbrace{\left[x - \frac{x^5}{5 \cdot 4} + \frac{x^9}{9 \cdot 8 \cdot 5 \cdot 4} - \frac{x^{13}}{13 \cdot 12 \cdot 9 \cdot 8 \cdot 5 \cdot 4} + \cdots\right]}_{y_1(x)}$$

Two linearly independent solutions are the functions y_0, y_1 above.

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Problem 2: Consider the differential equation $2xy'' + y' - 4y = 0$.

(a) Show that $x = 0$ is a regular singular point of this equation.

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$y'' + P(x)y' + Q(x)y = 0$ with coefficients $P(x) = 1/(2x)$ and $Q(x) = -2/x$.

Both P and Q are singular (not analytic) at $x = 0$, so $x = 0$ is a singular point.

But we have that $xP(x) = 1/2$ and $x^2Q(x) = -2x$ are both analytic functions (polynomials) so $x = 0$ is a *regular* singular point.

(b) Find the values of r such that a solution of this equation can we written as the series $y = \sum_{n=0}^{\infty} a_n x^{n+r}$.

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$$y = \sum_{n=0}^{\infty} a_n x^{n+r}, \qquad y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}, \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{n+r-2}.$$

$$\text{(sub into DE)} \implies \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} 4a_n x^{n+r} = 0$$

$$\text{(re-index)} \implies \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1} - \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r-1} = 0$$

Pull out leading $(n = 0)$ terms:

$$\implies \big[2r(r-1) + r\big]a_0 x^{r-1} + \sum_{n=1}^{\infty} \big[2(n+r)(n+r-1)a_n + (n+r)a_n - 4a_{n-1}\big]x^{n+r-1} = 0$$

Leading term gives the "indicial equation":

$$0 = 2r(r-1) + r = 2r^2 - r = r(2r-1) \implies \boxed{r = 0 \text{ or } r = \frac{1}{2}}$$

(c) For each value of r, give a (simplified) recurrence relation for the coefficients a_n in the series above. You do **not** need to solve this recurrence relation or find a formula for $y(x)$.

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$$0 = 2(n+r)(n+r-1)a_n + (n+r)a_n - 4a_{n-1} = (2n+2r-1)(n+r)a_n - 4a_{n-1}$$

case $r = 0$:

$$0 = (2n-1)na_n - 4a_{n-1} \implies \boxed{a_n = \frac{4a_{n-1}}{(2n-1)n} = \frac{2a_{n-1}}{n(n-\frac{1}{2})} \text{ or } a_{n+1} = \frac{4a_n}{(2n+1)(n+1)}}$$

case $r = \frac{1}{2}$:

$$0 = 2n(n+\tfrac{1}{2})a_n - 4a_{n-1} \implies \boxed{a_n = \frac{a_{n-1}}{n(n+\frac{1}{2})} = \frac{4a_{n-1}}{n(2n+1)} \text{ or } a_{n+1} = \frac{4a_n}{(n+1)(2n+3)}}$$

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Problem 3: Use the Laplace transform to solve the following initial value problems for $y(t)$:

(a) $y'' = e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1$

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$$s^2Y - 0 \cdot s - 1 = \frac{1}{s+2} \implies s^2Y = 1 + \frac{1}{s+2} \implies Y(s) = \frac{1}{s^2} + \frac{1}{s^2(s+2)}$$

Partial fractions:

$$\frac{1}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} = \frac{As(s+2) + B(s+2) + Cs^2}{s^2(s+2)}$$

$$s = -2: \quad 4C = 1 \implies C = 1/4$$
$$s = 0: \quad 2B = 1 \implies B = 1/2$$
$$\left.\frac{d}{ds}\right|_{s=0}: \quad 2A + B = 0 \implies A = -B/2 = -1/4$$

$$\implies y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s+2}\right\}$$
$$= t - \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}e^{-2t}$$
$$= \boxed{\frac{3}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2t}}$$

(b) $y'' + 2y' + y = \delta(t-1), \quad y(0) = y'(0) = 0$

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$$s^2Y + 2sY + Y = e^{-s} \implies Y(s) = \frac{e^{-s}}{(s+1)^2}$$

$$\implies y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} = u(t-1)f(t-1)$$

where $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^{-t}$

$$\implies \boxed{y(t) = u(t-1)(t-1)e^{-(t-1)}}$$

/5 **Problem 4:** Find $\mathcal{L}^{-1}\left\{\dfrac{5-2s}{s^2-2s+5}\right\}$ where $\mathcal{L}$ denotes the Laplace transform.

Complete the square:

$$F(s) = \frac{5-2s}{s^2-2s+5} = \frac{5-2s}{(s-1)^2+4} = \frac{3-2(s-1)}{(s-1)^2+4} = \frac{3}{2}\cdot\frac{2}{(s-1)^2+4} - 2\cdot\frac{(s-1)}{(s-1)^2+4}$$

$$\begin{aligned}\implies f(t) = \mathcal{L}^{-1}\{F(s)\} &= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\} - 2\mathcal{L}^{-1}\left\{\frac{(s-1)}{(s-1)^2+4}\right\}\\ &= \frac{3}{2}e^t\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} - 2e^t\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}\\ &= \boxed{\frac{3}{2}e^t\sin(2t) - 2e^t\cos(2t)}\end{aligned}$$

/5 **Problem 5:** Use the *definition* of the Laplace transform $\mathcal{L}$ to find $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2. \end{cases}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)\,dt\\ &= \int_0^2 te^{-st}\,dt + \int_2^\infty 2e^{-st}\,dt \qquad \begin{cases} u = t & dv = e^{-st}\,dt \\ du = dt & v = -\frac{1}{s}e^{-st}\end{cases}\\ &= \left[-\frac{t}{s}e^{-st}\right]_{t=0}^2 + \int_0^2 \frac{1}{s}e^{-st} + \int_2^\infty 2e^{-st}\,dt\\ &= -\frac{2}{s}e^{-2s} - \left[\frac{1}{s^2}e^{-st}\right]_{t=0}^2 - \left[\frac{2}{s}e^{-st}\right]_{t=2}^\infty\\ &= -\frac{2}{s}e^{-2s} - \frac{1}{s^2}e^{-2s} + \frac{1}{s^2} + \frac{2}{s}e^{-2s}\\ &= \boxed{\frac{1}{s^2}\left(1-e^{-2s}\right)}\end{aligned}$$