THOMPSON RIVERS UNIVERSITY

MATH 3170
Calculus 4

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MIDTERM EXAM #2
SOLUTIONS

25 March 2013   13:30–14:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
TOTAL:		30

/10 **Problem 1:** Evaluate $\iint_S \mathbf{F}\cdot\hat{n}\,dS$ where

$$\mathbf{F} = ze^{x^2}\mathbf{i} + 3y\mathbf{j} + (2 - yz^7)\mathbf{k}$$

and S is the union of the five "upper" faces of the unit cube $[0,1]\times[0,1]\times[0,1]$. That is, the $z=0$ face is *not* part of S.

This is a good one for the Divergence Theorem, but we need to form a closed surface, say by adding the bottom face $S' = [0,1]\times[0,1]\times\{0\}$:

$$\iint_S \mathbf{F}\cdot\hat{n}\,dS + \underbrace{\iint_{S'} \mathbf{F}\cdot\hat{n}\,dS}_{\int_{S'}(0,3y,2)\cdot(0,0,-1)\,dA = -2} = \oint_{S\cup S'} \mathbf{F}\cdot\hat{n}\,dS$$

$$= \iint_D \nabla\cdot\mathbf{F}\,dV \qquad (D = [0,1]\times[0,1]\times[0,1])$$

$$= \int_0^1\int_0^1\int_0^1 (2xze^{x^2} + 3 - 7yz^6)\,dx\,dy\,dz$$

$$= \underbrace{\int_0^1 z\,dz}_{1/2}\underbrace{\int_0^1 dy}_{1}\underbrace{\int_0^1 2xe^{x^2}\,dx}_{e^{x^2}\big|_0^1 = e-1} + 3 - \underbrace{\int_0^1 dx}_{1}\underbrace{\int_0^1 y\,dy}_{1/2}\underbrace{\int_0^1 7z^6\,dz}_{1}$$

$$= \frac{e}{2} + 2$$

$$\implies \int_S \mathbf{F}\cdot\hat{n}\,dS = \left(\frac{e}{2}+2\right) + 2 = \boxed{\frac{e}{2}+4}$$

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Problem 2: Suppose $\mathbf{F}(x, y, z)$ is a continuously differentiable vector field, and that $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere. Prove that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for *every* smooth closed curve C.

Since $\mathbf{F}$ is continuously differentiable and C is smooth, by Stokes's Theorem we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$$

where C is any piecewise-smooth surface whose boundary is C. But $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere, so regardless of the choice of S we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathbf{0} \cdot \hat{\mathbf{n}}\, dS$$

$$= \iint_S 0\, dS = 0.$$

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Problem 3: Let C be the triangular path with vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$, oriented counter-clockwise when viewed from the positive z-axis. Use Stokes's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = x^4\mathbf{i} + xy\mathbf{j} + z^4\mathbf{k}.$$

Since C is piecewise-smooth and $\mathbf{F}$ is continuously differentiable, we can apply Stokes's Theorem:

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS.$$

We have

$$\nabla\times\mathbf{F} = (0,0,y)$$

and for S we can choose any piecewise-smooth surface whose boundary is C. Let's take S to be the triangle with the given vertices. The triangle lies is part of the plane $z = 2-x-y$, or in parametrized form:

$$\begin{cases} x = x \\ y = y \\ z = 2-x-y \end{cases} \implies \mathbf{r}(x,y) = (x,y,2-x-y) \implies \begin{array}{l} \mathbf{r}_x = (1,0,-1) \\ \mathbf{r}_y = (0,1,-1). \end{array}$$

This gives

$$\begin{aligned}
\iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS &= \iint_D (\nabla\times\mathbf{F})\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dx\,dy \qquad (D = \{(x,y) : 0\le y\le 2-x,\ 0\le x\le 2\}) \\
&= \iint_D (0,0,y)\cdot(1,1,1)\,dx\,dy \\
&= \iint_D y\,dx\,dy \\
&= \int_0^2\int_0^{2-x} y\,dy\,dx \\
&= \int_0^2 \frac{1}{2}(2-x)^2\,dx = -\frac{1}{6}(2-x)^3\Big|_0^2 = \boxed{\frac{4}{3}}
\end{aligned}$$