THOMPSON RIVERS UNIVERSITY

MATH 3170
Calculus 4

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

27 February 2013 13:30–14:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
TOTAL:		40

/10

Problem 1: The field $\mathbf{F}(x,y,z) = (axy+z)\mathbf{i} + x^2\mathbf{j} + (bx+2z)\mathbf{k}$ is conservative.

(a) Determine the values of a and b.

For a conservative field we have

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy+z & x^2 & bx+2z \end{vmatrix} = \big(0, 1-b, (2-a)x\big)$$

$$\Longrightarrow \boxed{a=2,\ b=1}$$

(b) Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where C is the unit circle.

The field is conservative, so the integral around *any* closed path is 0.

(c) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve from $(1,1,0)$ to $(0,0,3)$ that lies on the intersection of the surfaces $2x+y+z=3$ and $9x^2+9y^2+2z^2=18$ in the first octant.

First find the potential function f (so that $F = \nabla f$):

$$\frac{\partial f}{\partial x} = 2xy+z \qquad \Longrightarrow \ f = x^2y + xz + C(y,z)$$

$$\frac{\partial f}{\partial y} = x^2 = x^2 + \frac{\partial C}{\partial y} \Longrightarrow \frac{\partial C}{\partial y} = 0 \qquad \Longrightarrow \ f = x^2y + xz + C(z)$$

$$\frac{\partial f}{\partial z} = x + 2z = x + C'(z) \Longrightarrow C'(z) = 2z \qquad \Longrightarrow \ f = x^2y + xz + z^2$$

Now we have that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = f(0,0,3) - f(1,1,0)$$

$$= 9 - 1 = \boxed{8}$$

/10 **Problem 2:** Find $\iiint_B (x^2+y^2)\,dV$ where B is the ball given by $x^2+y^2+z^2 \le a^2$.

In polar coordinates $x^2+y^2=(r\sin\phi)^2$ and $dV = r^2\sin\phi\,dr\,d\theta\,d\phi$ so the integral becomes

$$\int_0^\pi \int_0^{2\pi} \int_0^a (r\sin\phi)^2\, r^2 \sin\phi\,dr\,d\theta\,d\phi = \underbrace{\int_0^a r^4\,dr}_{\frac{1}{5}a^5} \cdot \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \cdot \underbrace{\int_0^\pi \sin^3\phi\,d\phi}_{\frac{4}{3}} = \boxed{\frac{8\pi a^5}{15}}$$

Here we've used

$$\begin{aligned}\int_0^\pi \sin^3\phi\,d\phi &= \int_0^\pi (1-\cos^2\phi)\sin\phi\,d\phi \\ &= \int_{-1}^1 (1-u^2)\,du \qquad (u=\cos\phi;\ du=-\sin\phi\,d\phi) \\ &= \left[u-\frac{1}{3}u^3\right]_{-1}^1 = \frac{2}{3}-\left(-\frac{2}{3}\right)=\frac{4}{3}\end{aligned}$$

/10 **Problem 3:** Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = (x^2y^2, x^3y)$$

and the path C is counter-clockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

method 1 (Green's Thm): Let $P = x^2y^2$, $Q = x^3y$ and let D be the given unit square.

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
&= \iint_D (3x^2y - 2x^2y)\, dA = \iint_D x^2y\, dA \\
&= \int_0^1 \int_0^1 x^2y\, dx\, dy \\
&= \underbrace{\int_0^1 x^2\, dx}_{\frac{1}{3}} \cdot \underbrace{\int_0^1 y\, dy}_{\frac{1}{2}} = \boxed{\frac{1}{6}}
\end{aligned}$$

method 2 (direct approach): break the integral up into four integrals along the individual line segments:

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C x^2y^2\, dx + x^3y\, dy \\
&= \underbrace{\int_0^1 x^2(0)^2\, dx}_{0} + \underbrace{\int_0^1 (1)^3y\, dy}_{\frac{1}{2}} + \underbrace{\int_1^0 x^2(1)^2\, dx}_{-\frac{1}{3}} + \underbrace{\int_1^0 (0)^3y\, dy}_{0} \\
&= \boxed{\frac{1}{6}}
\end{aligned}$$

/10

Problem 4: Let A be the area of the part of the surface $z = \frac{1}{2}x^2$ in first octant that lies within the cylinder $x^2 + y^2 \leq 1$. Show that

$$A = \frac{1}{3}\int_0^{\pi/2} \frac{(1+\cos^2\theta)^{3/2} - 1}{\cos^2\theta}\, d\theta.$$

(Do not attempt to evaluate this integral; the answer can't be expressed in terms of elementary functions. It turns out the answer can be written as $\frac{\sqrt{2}}{3}K(\frac{1}{2})$ where $K(x)$ is the "complete elliptic integral of the first kind".)

first parametrize the surface:

$$\begin{cases} x = x \\ y = y \\ z = \frac{1}{2}x^2 \end{cases} \implies \begin{matrix} \mathbf{r}_x = (1, 0, x) \\ \mathbf{r}_y = (0, 1, 0) \end{matrix} \implies \mathbf{r}_x \times \mathbf{r}_y = (-x, 0, 1)$$

then:

$$A = \iint_D |\mathbf{r}_x \times \mathbf{r}_y|\, dx\, dy \qquad (D \text{ is the first-quadrant quarter of the unit circle})$$

$$= \iint_D \sqrt{1+x^2}\, dx\, dy$$

$$= \int_0^{\pi/2}\int_0^1 \sqrt{1+(r\cos\theta)^2}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_1^{1+\cos^2\theta} \sqrt{u}\cdot\frac{du}{2\cos^2\theta}\, d\theta \qquad (u = 1 + r^2\cos^2\theta;\ du = 2r\cos^2\theta\, dr)$$

$$= \int_0^{\pi/2} \frac{2}{3}u^{3/2}\Big|_1^{1+\cos^2\theta}\cdot\frac{1}{2\cos^2\theta}\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{(1+\cos^2\theta)^{3/2} - 1}{\cos^2\theta}\, d\theta$$