THOMPSON RIVERS UNIVERSITY

MATH 3170
Calculus 4

Instructor: Richard Taylor

FINAL EXAM
SOLUTIONS

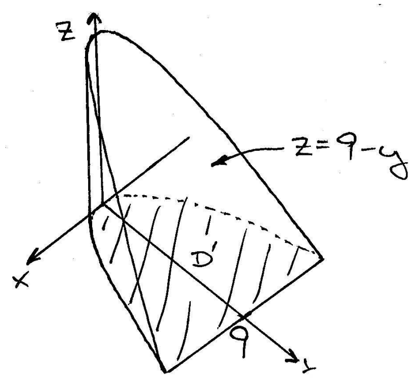
16 April 2013   09:00–12:00

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
5		6
6		10
7		10
8		10
9		10
TOTAL:		86

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 10 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

/10 **Problem 1:** Evaluate $\iiint_D 8xyz\,dV$ where D is the region bounded by the surface $y = x^2$, the plane $y + z = 9$, and the xy-plane.



We can write this as an iterated integral:

$$\iint_{D'} \int_0^{9-y} 8xyz\,dz\,dA$$

where D' is the region in the xy-plane bounded by $y = x^2$ and $y = 9$...

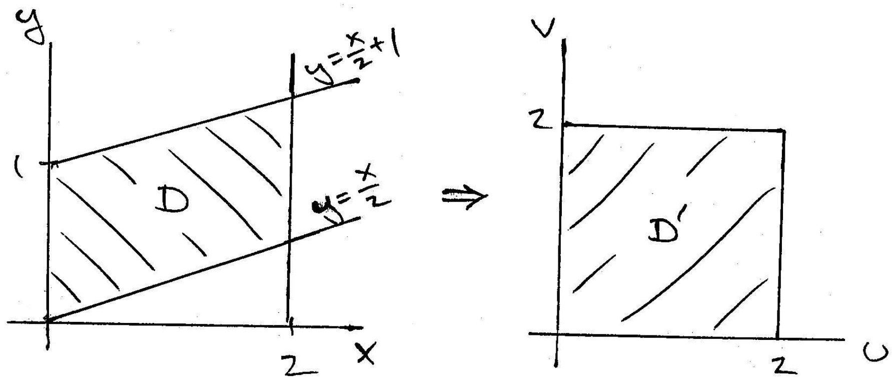
$$= \int_{-3}^{3} \int_{x^2}^{9} \int_0^{9-y} 8xyz\,dz\,dy\,dx$$

which, after doing the integrals, evaluates to 0.

We might have anticipated this answer using symmetry: D is symmetric about the yz-plane; the integrand is also symmetric, and odd in x.

/10 **Problem 2:** Evaluate $\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx$ using the substitution $u = x$, $v = 2y - x$.

In the xy-plane the region of integration is a parallelogram with boundaries $x = 0$, $x = 2$, $y = x/2$, $y = x/2 + 1$.



In the uv-plane the boundary $y = x/2$ transforms to $v = 0$. The boundary $y = x/2+1$ becomes $v = 2$. So,

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2y-x)e^{(2y-x)^2}\,dy\,dx = \iint_D u^5 v e^{v^2}\, J(u,v)\,dA$$

where D is the square $[0,2]\times[0,2]$ and the Jacobian J is given by

$$J = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}^{-1} = \begin{vmatrix} 1 & 0 \\ -1 & 2 \end{vmatrix}^{-1} = 1/2.$$

Thus the double integral can be written as the iterated integral

$$\int_0^2 \int_0^2 u^5 v e^{v^2} \cdot \frac{1}{2}\,du\,dv = \underbrace{\int_0^2 u^5\,du}_{\frac{1}{6}2^6}\ \underbrace{\int_0^2 \frac{1}{2} v e^{v^2}\,dv}_{\frac{1}{4}e^{v^2}\big|_0^2 = \frac{1}{4}(e^4-1)} = \boxed{\frac{8}{3}(e^4-1)}$$

/10 **Problem 3:** Evaluate $\int_C yz\,ds$ where C is the curve of intersection of the surfaces $y = \cos x$ and $z = \sin x$, from $(0, 1, 0)$ to $(\pi, -1, 0)$.

Parametrize C:

$$\begin{cases} x = x \\ y = \cos x \\ z = \sin x \end{cases} \implies \mathbf{r}(x) = (x, \cos x, \sin x)$$

$$\implies ds = |\mathbf{r}'(x)|\,dx = |(1, -\sin x, \cos x)|\,dx = \sqrt{1 + \sin^2 x + \cos^2 x}\,dx = \sqrt{2}\,dx$$

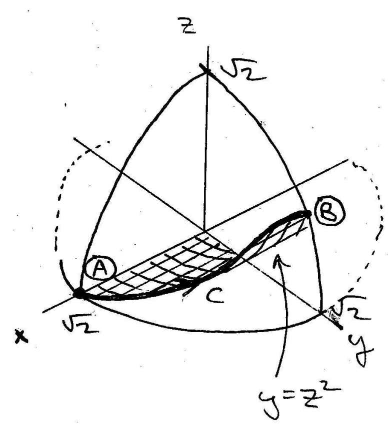
$$\implies \int_C yz\,ds = \int_0^\pi \cos x \cdot \sin x \cdot \sqrt{2}\,dx$$

$$= \sqrt{2} \cdot \frac{1}{2} \sin^2 x \Big|_0^\pi = \boxed{0}$$

/10 **Problem 4:** Let $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} + \cos y \sin z\,\mathbf{j} + \sin y \cos z\,\mathbf{k}$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{s}$ where C is the intersection, in the first octant, of the surface $z = y^2$ with the sphere $x^2+y^2+z^2 = 2$ (oriented counter-clockwise when viewed from the positize z-axis.

A straightforward calculation gives $\nabla\times\mathbf{F} = \mathbf{0}$, so the field is conservative. Thus $\mathbf{F} = \nabla f$ where

$$f_x = x^2 \implies f = \frac{1}{3}x^3 + C(y,z)$$
$$f_y = \cos y \sin z = \frac{\partial C}{\partial y} \implies f = \frac{1}{3}x^3 + \sin y \sin z + C(z)$$
$$f_z = \sin y \cos z = \sin y \cos z + C'(z) \implies f = \frac{1}{3}x^3 + \sin y \sin z.$$



In the xz-plane the surfaces intersect at A, where

$$\begin{cases} z = 0^2 = 0 \\ x^2 + 0^2 + z^2 = 2 \end{cases} \implies A = (\sqrt{2},0,0)$$

and in the yz-plane they intersect at B, where

$$\begin{cases} z = y^2 \\ 0^2 + y^2 + z^2 = 2 \end{cases} \implies z^2 + z - 2 = 0 \implies z = 1 \implies B = (0,1,1).$$

Thus,

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{s} &= f(B) - f(A) \\ &= f(0,1,1) - f(\sqrt{2},0,0) \\ &= \boxed{\sin^2 1 - \frac{2^{3/2}}{3}} \end{aligned}$$

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Problem 5: Suppose C is a piecewise-smooth, simple closed curve in the xy-plane. Let f and g be continuously differentiable single-variable functions. Prove that

$$\oint_C f(x)\,dx + g(y)\,dy = 0.$$

Since f and g satisfy the hypotheses of Green's Theorem, we have

$$\begin{aligned}\oint_C f(x)\,dx + g(y)\,dy &= \iint_D \left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)\,dA \\ &= \iint_D (0-0)\,dA \\ &= 0\end{aligned}$$

where D is the plane region enclosed by C.

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Problem 6: Let S denote the surface of the cylinder $x^2+y^2=4$, $-2 \le z \le 2$, and consider the surface integral

$$\int_S (z - x^2 - y^2)\, dS$$

(a) Use an appropriate parametrization of S to calculate the value of the integral.

Parametrize S:

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \\ z = z \end{cases} \implies \mathbf{r}(\theta, z) = (-2\sin\theta, 2\cos\theta, z)$$

$$\implies \begin{matrix} \mathbf{r}_\theta = (-2\sin\theta, 2\cos,\theta, 0) \\ \mathbf{r}_z = (0,0,1) \end{matrix} \implies dS = |\mathbf{r}_\theta \times \mathbf{r}_z|\, d\theta\, dz = |(2\cos\theta, 2\sin\theta, 0)|\, d\theta\, dz = 2\, d\theta\, dz$$

Thus,

$$\begin{aligned} \int_S (z - x^2 - y^2)\, dS &= \int_{-2}^{2}\int_0^{2\pi} (z-4)\, 2\, d\theta\, dz \\ &= 2 \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_{-2}^{2} (z-4)\, dz}_{\frac{1}{2}z^2 - 4z\Big|_{-2}^{2} = -16} = \boxed{-64\pi} \end{aligned}$$

(b) Use symmetry to evaluate the integral without resorting to a parametrization of the surface.

We have

$$\begin{aligned} \int_S (z - x^2 - y^2)\, dS &= \int_S (z-4)\, dS \\ &= \underbrace{\int_S z\, dS}_{\text{0 by symmetry}} - 4 \underbrace{\int_S dS}_{\text{area of } S} \\ &= 0 - 4(2\pi)(2)(4) = -64\pi \end{aligned}$$

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Problem 7: Let S be the "silo" composed of the union of surfaces S_1, S_2 where S_1 is the cylinder

$$S_1: \quad x^2+y^2=9, \quad 0 \le z \le 8$$

and S_2 is the hemisphere

$$S_2: \quad x^2+y^2+(z-8)^2=9. \quad z \ge 8.$$

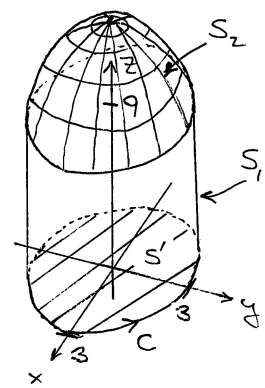
Considering S to have its normal oriented away from the origin, evaluate $\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$ where

$$\mathbf{F}(x,y,z)=(x^3+xz+yz^2)\,\mathbf{i}+(xyz^3+y^7)\,\mathbf{j}+x^2z^5\,\mathbf{k}.$$

Since S is piecewise-smooth we can use Stokes' Theorem:

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}=\oint_C \mathbf{F}\cdot d\mathbf{r}$$

where C is the boundary of S, i.e. the circle of radius 3 in the xy-plane, centered at the origin.



Parametrize C:

$$\begin{cases} x=3\cos\theta \\ y=3\sin\theta \\ z=0 \end{cases} \implies \mathbf{r}(\theta)=(3\cos\theta,3\sin\theta,0)$$

$$\implies d\mathbf{r}=\mathbf{r}'(\theta)\,d\theta=(-3\sin\theta,3\cos\theta,0)\,d\theta$$

Thus,

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(\cos^3\theta,\sin^7\theta,0)\cdot(-3\sin\theta,3\cos\theta,0)\,d\theta$$
$$=\int_0^{2\pi}(-\sin\theta\cos^3\theta+\cos\theta\sin^7\theta)\,d\theta=\boxed{0}$$

Alternatively, we can use Stokes' Theorem to trade the given integral over S for an integral over any other surface S' that has the same boundary C:

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}=\iint_{S'}\nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

We have

$$\nabla\times\mathbf{F}=(-3xyz^2,x+2yz-2xz^5,yz^3-z^2).$$

Thus if we take S' to be the circle in the xy-plane enclosed by C we get

$$\iint_{S'}\nabla\times\mathbf{F}\cdot d\mathbf{S}=\iint_{S'}(0,x,0)\cdot\mathbf{k}\,dA=\iint_{S'}0\,dA=\boxed{0}$$

/10 **Problem 8:** Let $\mathbf{F}(x, y, z) = 2xy\,\mathbf{i} + y^2\,\mathbf{i} + 3yz\,\mathbf{k}$. Calculate the total flux $\oint_S \mathbf{F} \cdot d\mathbf{S}$ where S is

(a) the ball $x^2 + y^2 + z^2 = a^2$.

We have

$$\nabla \cdot \mathbf{F} = 2y + 2y + 3y = 7y$$

so applying the Divergence Theorem gives

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \boxed{0} \quad \text{(by symmetry)}$$

(b) the cube $[0, a] \times [0, a] \times [0, a]$.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \underbrace{\int_0^a dx}_{a} \underbrace{\int_0^a 7y\, dy}_{\frac{7}{2}a^2} \underbrace{\int_0^a dz}_{a} = \boxed{\frac{7}{2}a^4}$$

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Problem 9: Use tensor notation to prove the vector identity

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}.$$

We have

$$(\mathbf{a}\times\mathbf{b})_i = \varepsilon_{ijk}a_jb_k$$

so that

$$\begin{aligned}
((\mathbf{a}\times\mathbf{b})\times\mathbf{c})_n &= \varepsilon_{nim}(\varepsilon_{ijk}a_jb_k)c_m \\
&= \varepsilon_{nim}\varepsilon_{ijk}a_jb_kc_m \\
&= \varepsilon_{imn}\varepsilon_{ijk}a_jb_kc_m \quad \text{(even permutation)} \\
&= (\delta_{mj}\delta_{nk} - \delta_{mk}\delta_{nj})a_jb_kc_m \\
&= \delta_{mj}\delta_{nk}a_jb_kc_m - \delta_{mk}\delta_{nj}a_jb_kc_m \\
&= a_jb_nc_j - a_nb_kc_k \\
&= \underbrace{(a_jc_j)}_{\mathbf{a}\cdot\mathbf{c}}b_n - \underbrace{(b_kc_k)}_{\mathbf{b}\cdot\mathbf{c}}a_n \\
&= ((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a})_n \quad \checkmark
\end{aligned}$$