# MATH 2670: Quiz #7 – SOLUTIONS

/10 **Problem 1:** Find two power series solutions (centered about $x = 0$) for the following differential equation:

$$y'' + x^2 y = 0$$

Assume a power series centered about $x = 0$:

$$y = \sum_{n=0}^{\infty} c_n x^n \qquad y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitute into the DE:

$$\begin{aligned}
0 &= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x^2 \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+2} \qquad \text{re-index to get like powers} \ldots \\
&= \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=4}^{\infty} c_{n-4} x^{n-2} \qquad \text{pull out extra terms} \ldots \\
&= \underbrace{2c_2 + 6c_3 x}_{n=2,\,3} + \sum_{n=4}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=4}^{\infty} c_{n-4} x^{n-2} \qquad \text{combine series} \ldots \\
&= 2c_2 + 6c_3 x + \sum_{n=4}^{\infty} \big[n(n-1)c_n + c_{n-4}\big] x^{n-2}.
\end{aligned}$$

All coefficients in this power series must be zero. This gives the recurrence relations:

$$\begin{cases} c_2 = c_3 = 0 \\ n(n-1)c_n + c_{n-4} = 0 \implies c_n = -\dfrac{1}{n(n-1)} c_{n-4} \quad (n = 4, 5, 6, \ldots) \end{cases}$$

Solve the recurrence relation:

$$\begin{aligned}
n = 4: &\quad c_4 = -\tfrac{1}{4\cdot 3} c_0 \\
n = 5: &\quad c_5 = -\tfrac{1}{5\cdot 4} c_1 \\
n = 6: &\quad c_6 = -\tfrac{1}{6\cdot 5} c_2 = 0 \quad (= c_{10} = c_{14} = \cdots) \\
n = 7: &\quad c_7 = -\tfrac{1}{7\cdot 6} c_3 = 0 \quad (= c_{11} = c_{15} = \cdots) \\
n = 8: &\quad c_8 = -\tfrac{1}{8\cdot 7} c_4 = \tfrac{1}{8\cdot 7\cdot 4\cdot 3} c_0 \\
n = 9: &\quad c_9 = -\tfrac{1}{9\cdot 8} c_5 = \tfrac{1}{9\cdot 8\cdot 5\cdot 4} c_1 \\
&\quad \vdots
\end{aligned}$$

The pattern is fairly clear and we now have the solution

$$\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
&= c_0 + c_1 x - \tfrac{1}{4\cdot 3} c_0 x^4 - \tfrac{1}{5\cdot 4} c_1 x^5 + \tfrac{1}{8\cdot 7\cdot 4\cdot 3} c_0 x^8 + \tfrac{1}{9\cdot 8\cdot 5\cdot 4} c_1 x^9 + \cdots \\
&= c_0 \left(1 - \tfrac{1}{4\cdot 3} x^4 + \tfrac{1}{8\cdot 7\cdot 4\cdot 3} x^8 - \cdots\right) + c_1 \left(x - \tfrac{1}{5\cdot 4} x^5 + \tfrac{1}{9\cdot 8\cdot 5\cdot 4} x^9 - \cdots\right).
\end{aligned}$$

We can now identify two independent solutions (by setting $c_0$ or $c_1$ equal to 0):

$$\boxed{\begin{aligned}
y_0(x) &= 1 - \tfrac{1}{4\cdot 3} x^4 + \tfrac{1}{8\cdot 7\cdot 4\cdot 3} x^8 - \tfrac{1}{12\cdot 11\cdot 8\cdot 7\cdot 4\cdot 3} x^{12} + \cdots \\
y_1(x) &= x - \tfrac{1}{5\cdot 4} x^5 + \tfrac{1}{9\cdot 8\cdot 5\cdot 4} x^9 - \tfrac{1}{13\cdot 12\cdot 9\cdot 8\cdot 5\cdot 4} x^{13} + \cdots
\end{aligned}}$$