MATH 2670: Quiz #6 – SOLUTIONS

/5 **Problem 1:** Use the Laplace transform to solve the following initial-value problem:

$$\begin{cases} y'' - 2y' = \delta(t-2) \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Applying the Laplace transform gives

$$\left[s^2Y - s\cdot 0 - 1\right] - 2\left[sY - 0\right] = e^{-2s} \implies Y(s) = \frac{e^{-2s}+1}{s^2-2s} = \frac{e^{-2s}+1}{s(s-2)}$$

Expand by partial fractions:

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} = \frac{A(s-2)+Bs}{s(s-2)} \qquad \begin{array}{lll} s=0: & 1=-2A & \implies A=-\frac{1}{2} \\ s=2: & 1=2B & \implies B=\frac{1}{2} \end{array}$$

Invert the Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}+1}{s(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s-2)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = u(t-2)f(t-2) + f(t)$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{-1/2}{s} + \frac{1/2}{s-2}\right\} = -\tfrac{1}{2}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{1} + \tfrac{1}{2}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}}_{e^{2t}} = -\frac{1}{2} + \frac{1}{2}e^{2t}$$

so that

$$\boxed{y(t) = u(t-2)\left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right] + \frac{1}{2}e^{2t} - \frac{1}{2}}$$

/5 **Problem 2:** Find a Fourier series representation of the function $f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 \le x < \pi. \end{cases}$

Extending this to a periodic function with period is $2p = 2\pi$ gives $p = \pi$. Then the Fourier series is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \end{aligned}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\cos(nx)}_{\text{odd function}}\, dx = 0$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{even function}}\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\, dx \\ &= \frac{2}{\pi}\int_0^{\pi} \sin(nx)\, dx = \frac{2}{\pi}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi} = \frac{2}{n\pi}\left[1 - \underbrace{\cos(n\pi)}_{(-1)^n}\right] \end{aligned}$$

so that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) = \boxed{\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1-(-1)^n\right]\sin(nx)}$$