MATH 2670: Quiz #4 – SOLUTIONS

/3 **Problem 1:** Find the general solution $y(x)$ of the differential equation $(\cos x)y' + (\sin x)y = 1$.

Re-write the DE in "standard form" as

$$y' + \underbrace{\frac{\sin x}{\cos x}}_{p(x)} y = \frac{1}{\cos x}$$

Multiply both sides by the integrating factor

$$e^{\int p(x)\,dx} = e^{\int \frac{\sin x}{\cos x}\,dx} = e^{-\ln(\cos x)} = e^{\ln([\cos x]^{-1})} = \frac{1}{\cos x}$$

$$\implies \underbrace{\frac{1}{\cos x}y' + \frac{\sin x}{\cos^2 x}y}_{\frac{d}{dx}\frac{y}{\cos x}} = \frac{1}{\cos^2 x} \ \left(= \sec^2 x\right)$$

Integrate both sides:

$$\implies \frac{y}{\cos x} = \int \sec^2 x\,dx = \tan x + C \implies \boxed{y = \sin x + C\cos x, \quad C \in \mathbb{R}}$$

/3 **Problem 2:** Find the general solution $y(x)$ of the differential equation $\frac{dy}{dx} + 2xy^2 = 0$.

This equation is nonlinear, but at least separable:

$$\frac{dy}{dx} = -2xy^2 \implies \int y^{-2}\,dy = \int -2x\,dx$$

$$\implies -y^{-1} = x^2 + C$$

$$\implies \boxed{y = \frac{1}{x^2 + C}, \quad C \in \mathbb{R}}$$

/4 **Problem 3:** A tank initially contains 50 liters of pure water. Chlorinated water containing 5 mg of chlorine per liter is then pumped into the tank at a rate of 10 liters per minute. The well-mixed solution is pumped out at the same rate. How long does it take for the concentration of chlorine in the tank to reach 4 mg per liter?

Let $y(t)$ = mg of Cl in the tank after t minutes. Then

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out}) = 5 \times 10 - \frac{y}{50} \times 10$$

$$\implies \boxed{y' + 0.2y = 50} \quad (\text{a 1st-order linear DE})$$

Multiply both side by the integrating factor $\mu(t) = e^{\int 0.2\,dt} = e^{0.2t}$:

$$\implies \underbrace{e^{0.2t}y' + 0.2e^{0.2t}y}_{\frac{d}{dt}(e^{0.2t}y)} = 50e^{0.2t} \implies e^{0.2t}y = \int 50e^{0.2t}\,dt = 250e^{0.2t} + C$$

$$\implies y = 250 + Ce^{-0.2t}$$

The initial condition $y(0) = 0$ gives

$$0 = 250 + Ce^0 \implies C = -250 \implies y = 250(1 - e^{-0.2t}).$$

So, finally, we have $y(t) = 4 \times 50 = 200$ mg when

$$200 = 250 - 250e^{-0.2t} \implies e^{-0.2t} = \frac{50}{250} \implies \boxed{t = \frac{\ln 5}{0.2} \approx 8.04\,\text{min}}$$