MATH 2670: Quiz #3 – SOLUTIONS

/5 **Problem 1:** Evaluate the surface integral $\int_S xz^2\,dS$ where S is the part of the cone $y = \sqrt{x^2+z^2}$ above the xy-plane and between the planes $y = 0$ and $y = 5$.

Hint: you can parameterize S with the vector function $\mathbf{r}(R,\theta) = (R\cos\theta, R, R\sin\theta)$.

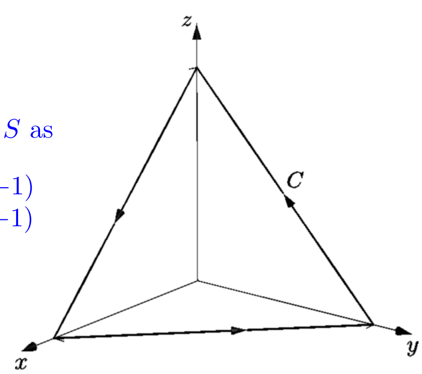
$$\mathbf{r}(R,\theta) = (R\cos\theta, R, R\sin\theta) \implies \begin{array}{l}\mathbf{r}_R = (\cos\theta, 1, \sin\theta)\\ \mathbf{r}_\theta = (-R\sin\theta, 0, R\cos\theta)\end{array} \qquad (\theta, R) \in D = [0,\pi]\times[0,5]$$

$$\mathbf{r}_R \times \mathbf{r}_\theta = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & 1 & \sin\theta \\ -R\sin\theta & 0 & R\cos\theta \end{vmatrix} = R\cos\theta\hat{\mathbf{i}} - R\hat{\mathbf{j}} + R\sin\theta\hat{\mathbf{k}}$$

$$\implies |\mathbf{r}_R \times \mathbf{r}_\theta| = \sqrt{(R\cos\theta)^2 + R^2 + (R\sin\theta)^2} = \sqrt{2}R$$

$$\begin{aligned}\implies \int_S xz^2\,dS &= \iint_D xz^2|\mathbf{r}_R \times \mathbf{r}_\theta|\,dA \\ &= \int_0^\pi \int_0^5 (R\cos\theta)(R\sin\theta)^2\sqrt{2}R\,dR\,d\theta \\ &= \sqrt{2}\underbrace{\int_0^\pi \sin^2\theta\cos\theta\,d\theta}_{\frac{1}{3}\sin^3\theta\big|_0^\pi = 0}\int_0^5 R^4\,dR = \boxed{0}\end{aligned}$$

/5 **Problem 2:** Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = (x+y)\hat{\mathbf{i}} + (y+z)\hat{\mathbf{j}} + (z+x)\hat{\mathbf{k}}$ and C is triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.



Let S be the triangular planar region bounded by C. We can parameterize S as

$$\mathbf{r}(x,y) = (x, y, 1-x-y) \implies \begin{array}{l}\mathbf{r}_x = (1,0,-1)\\ \mathbf{r}_y = (0,1,-1)\end{array}$$

where $(x,y) \in D =$ the triangular region in the xy-plane that lies below S. The surface normal consistent with the orientation of C is in the direction

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

We have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y & y+z & z+x \end{vmatrix} = -\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}.$$

Stokes' Theorem gives

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} &= \iint_D \nabla\times\mathbf{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA \\ &= \iint_D (-1,-1,-1)\cdot(1,1,1)\,dA = -3\underbrace{\iint_D dA}_{1/2} = \boxed{-\frac{3}{2}}\end{aligned}$$