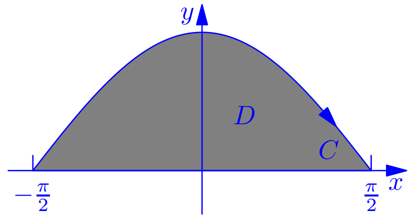
MATH 2670: Quiz #2 – SOLUTIONS

/5 **Problem 1:** Use Green's Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (e^{-x}+y^2)\hat{\mathbf{i}}+(e^{-y}+x^2)\hat{\mathbf{j}}$ and C consists of the arc of the curve $y=\cos x$ from $(-\frac{\pi}{2},0)$ to $(\frac{\pi}{2},0)$ and the line segment from $(\frac{\pi}{2},0)$ to $(-\frac{\pi}{2},0)$.



We have $\mathbf{F}(x,y) = P\hat{\mathbf{i}}+Q\hat{\mathbf{j}}$ where $P(x,y)=e^{-x}+y^2$ and $Q(x,y)=e^{-y}+x^2$. Green's Theorem gives

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} = -\oint_{-C}\mathbf{F}\cdot d\mathbf{r} = -\oint_{-C} P\,dx+Q\,dy &= -\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\,dA \\
&= -\iint_D (2x-2y)\,dA \\
&= -\underbrace{\iint_D 2x\,dA}_{\text{0 by symmetry}} + \iint_D 2y\,dA \\
&= \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} 2y\,dy\,dx \\
&= \int_{-\pi/2}^{\pi/2} \cos^2 x\,dx = \boxed{\frac{\pi}{2}}
\end{aligned}$$

/5 **Problem 2:** Find the area of the surface $z=\frac{2}{3}(x^{3/2}+y^{3/2})$ above the square where $0\le x\le 1$ and $0\le y\le 1$.

Parameterize the surface as $\mathbf{r}(u,v) = (u,v,\frac{2}{3}(u^{3/2}+v^{3/2}))$ with $(u,v)\in D=[0,1]\times[0,1]$. Then

$$\begin{matrix}\mathbf{r}_u = (1,0,u^{1/2}) \\ \mathbf{r}_v = (0,1,v^{1/2})\end{matrix} \implies \mathbf{r}_u\times\mathbf{r}_v = \begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & u^{1/2} \\ 0 & 1 & v^{1/2}\end{vmatrix} = -u^{1/2}\hat{\mathbf{i}}-v^{1/2}\hat{\mathbf{j}}+\hat{\mathbf{k}} \implies |\mathbf{r}_u\times\mathbf{r}_v| = \sqrt{u+v+1}$$

so

$$\begin{aligned}
A = \iint_S dS &= \iint_D |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv \\
&= \int_0^1\int_0^1 (u+v+1)^{1/2}\,du\,dv \\
&= \int_0^1 \tfrac{2}{3}(u+v+1)^{3/2}\Big|_{u=0}^{u=1}\,dv \\
&= \tfrac{2}{5}\left[\tfrac{2}{3}(u+v+1)^{5/2}\Big|_{u=0}^{u=1}\right]_{v=0}^{v=1} \\
&= \tfrac{4}{15}\left[(v+2)^{5/2}-(v+1)^{5/2}\right]_{v=0}^{v=1} \\
&= \tfrac{4}{15}\left(\left[3^{5/2}-2^{5/2}\right]-\left[2^{5/2}-1\right]\right) = \boxed{\tfrac{4}{15}\left(3^{5/2}-2\cdot 2^{5/2}+1\right)}
\end{aligned}$$