THOMPSON RIVERS UNIVERSITY

MATH 2670
Calculus 4 for Engineering

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MIDTERM EXAM #2 (IN-CLASS PORTION)
SOLUTIONS

27 March 2019 09:00–10:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		6
3		6
TOTAL:		18

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Problem 1: Consider the following differential equation for the function $y(x)$:

$$y'' + 4y' + 5y = 0.$$

(a) Find the general solution of this equation.

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$$y = e^{rx} \implies r^2 + 4r + 5 = 0 \implies r = -2 \pm i$$

$$\implies \boxed{y(x) = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x \quad (c_1, c_2 \in \mathbb{R})}$$

(b) Find the solution that satisfies the initial conditions $y(0) = 1$, $y'(0) = 0$.

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Impose the initial conditions on the solution from part (a):

$$y' = -2c_1 e^{-2x}\cos x - c_1 e^{-2x}\sin x - 2c_2 e^{-2x}\sin x + c_2 e^{-2x}\cos x$$

$$\begin{cases} y(0) = 1 = c_1 + 0 \implies c_1 = 1 \\ y'(0) = 0 = -2c_1 - 0 - 0 + c_2 \implies c_2 = 2c_1 = 2 \end{cases}$$

$$\implies \boxed{y(x) = e^{-2x}\cos x + 2e^{-2x}\sin x}$$

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Problem 2: Solve the following initial value problem:

$$y'' - 2y' + y = te^t + 4,\ y(0) = 1, y'(0) = 1.$$

homogeneous case:

$$y'' - 2y' + 4y = 0$$

$$y = e^{rt} \implies r^2 - 2r + 1 = 0 = (r-1)^2 \implies r = 1 \text{ (repeated root)}$$

$$\implies y_h(t) = c_1 e^t + c_2 t e^t \quad (c_1, c_2 \in \mathbb{R})$$

particular solution:

From the form of the rhs we can "guess" a particular solution of the form:

$$y(t) = (A + Bt)e^t + C = Ae^t + Bte^t + C$$

However, the first two terms are already solutions of the homogeneous case, so they won't work. Remedy this by multiplying these terms by t^2 (the lowest power of t required by make these terms independent of the homogeneous solutions):

$$\begin{aligned} y &= At^2e^t + Bt^3e^t + C \\ \implies y' &= 2Ate^t + At^2e^t + 3Bt^2e^t + Bt^3e^t \\ &= \big[2At + (A+3B)t^2 + Bt^3\big]e^t \\ \implies y'' &= \big[2A + 2(A+3B)t + 3Bt^2\big]e^t + \big[2At + (A+3B)t^2 + Bt^3\big]e^t \\ &= \big[2A + (4A+6B)t + (A+6B)t^2 + Bt^3\big]e^t \end{aligned}$$

Substitute into the DE:

$$\big[2A+(4A+6B)t+(A+6B)t^2+Bt^3\big]e^t - 2\big[2At+(A+3B)t^2+Bt^3\big]e^t + \big[At^2+Bt^3\big]e^t + C = te^t + 4$$

$$\implies 2Ae^t + 6Bte^t + C = te^t + 4 \implies A = 0,\ B = \tfrac{1}{6},\ C = 4$$

$$\implies y_p = \tfrac{1}{6}t^3e^t + 4$$

general solution:

$$y = y_h + y_p = c_1e^t + c_2te^t + \tfrac{1}{6}t^3e^t + 4$$

impose initial conditions:

$$y' = c_1e^t + c_2e^t + c_2te^t + \tfrac{1}{2}t^2e^t + \tfrac{1}{6}t^3e^t$$

$$\begin{cases} y(0) = 1 = c_1 + 4 \\ y'(0) = 1 = c_1 + c_2 \end{cases} \implies c_1 = -3,\ c_2 = 4$$

$$\implies \boxed{y = -3e^t + 4te^t + \tfrac{1}{6}t^3e^t + 4}$$

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Problem 3: Use the Laplace transform to solve the following initial value problem:

$$y'' - 2y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 0.$$

$$\big(s^2Y - sy(0) - y'(0)\big) - 2\big(sY - y(0)\big) + 4Y = 0$$

$$\implies \big(s^2Y - 2s\big) - 2\big(sY - 2\big) + 4Y = 0$$
$$\implies (s^2 - 2s + 4)Y - 2s + 4 = 0$$

$$\implies Y(s) = \frac{2s - 4}{s^2 - 2s + 4} = \frac{2(s-1) - 2}{(s-1)^2 + 3}$$

$$\implies y(t) = \mathcal{L}^{-1}\left\{\frac{2(s-1) - 2}{(s-1)^2 + 3}\right\}$$
$$= e^t\mathcal{L}^{-1}\left\{\frac{2s - 2}{s^2 + 3}\right\}$$
$$= e^t\mathcal{L}^{-1}\left\{2 \cdot \frac{s}{s^2 + (\sqrt{3})^2} - \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{s^2 + (\sqrt{3})^2}\right\}$$
$$= \boxed{e^t\left[2\cos(\sqrt{3}t) - \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)\right]}$$