THOMPSON RIVERS UNIVERSITY

MATH 2670
Calculus 4 for Engineering

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

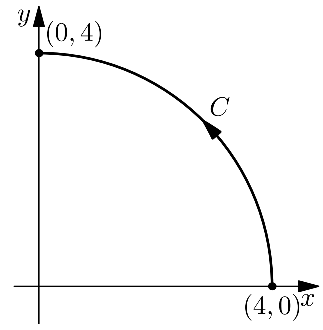
11 Feb. 2019 10:00–11:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		5
3		6
4		5
5		5
TOTAL:		26

/5 **Problem 1:** Evaluate the path integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \dfrac{1}{x^2+y^2}\hat{\mathbf{\imath}} + \dfrac{4}{x^2+y^2}\hat{\mathbf{\jmath}}$ and C is the quarter-circular arc shown below.



The arc has equation $x^2+y^2=16$, so on C the vector field simplifies as

$$\mathbf{F}(x,y) = \frac{1}{16}\hat{\mathbf{\imath}} + \frac{4}{16}\hat{\mathbf{\jmath}}.$$

Parameterizing C as $\mathbf{r}(\theta) = 4(\cos\theta, \sin\theta)$ $(0 \le \theta \le \frac{\pi}{2})$ gives

$$\mathbf{r}'(\theta) = 4(-\sin\theta, \cos\theta)$$

and so

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C \mathbf{F}(\mathbf{r}(\theta))\cdot\mathbf{r}'(\theta)\,d\theta \\
&= \int_0^{\pi/2} (\tfrac{1}{16}, \tfrac{1}{4})\cdot 4(-\sin\theta,\cos\theta)\,d\theta \\
&= \int_0^{\pi/2} (-\tfrac{1}{4}\sin\theta + \cos\theta)\,d\theta \\
&= -\tfrac{1}{4}\underbrace{\int_0^{\pi/2}\sin\theta\,d\theta}_{1} + \underbrace{\int_0^{\pi/2}\cos\theta\,d\theta}_{1} = 1 - \tfrac{1}{4} = \boxed{\frac{3}{4}}
\end{aligned}$$

/5 **Problem 2:** The vector field $\mathbf{F}(x,y) = 2xe^y\hat{\mathbf{\imath}} + (x^2e^y + y^2)\hat{\mathbf{\jmath}}$ is conservative (it is easy to check that $\nabla\times\mathbf{F} = \mathbf{0}$; you don't need to do this). Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is:

/3 (a) the graph of $y = x^2$ from the point $(0,0)$ to the point $(2,4)$.

Since $\mathbf{F}$ is conservative, let's seek a potential function $f(x,y)$ with $\nabla f = \mathbf{F}$:

$$\begin{aligned}
&\frac{\partial f}{\partial x} = 2xe^y \implies f(x,y) = x^2e^y + C(y) \\
\implies &\frac{\partial f}{\partial y} = x^2e^y + y^2 = x^2e^y + C'(y) \implies C'(y) = y^2 \implies C(y) = \tfrac{1}{3}y^3 \\
\implies &f(x,y) = x^2e^y + \tfrac{1}{3}y^3
\end{aligned}$$

Now we can use the fundamental theorem for line integrals:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,4) - f(0,0) = (2^2e^4 + \tfrac{1}{3}4^3) - (0+0) = \boxed{4e^4 + \frac{64}{3}}$$

/2 (b) the boundary of the unit square $[0,1]\times[0,1]$, with counter-clockwise orientation.

Since $\mathbf{F}$ is conservative and C is a closed loop:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \boxed{0}.$$

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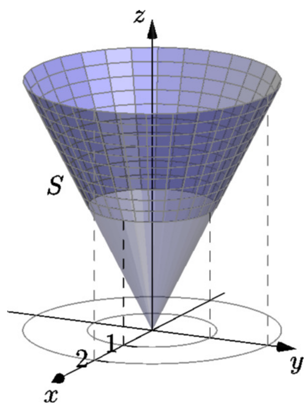
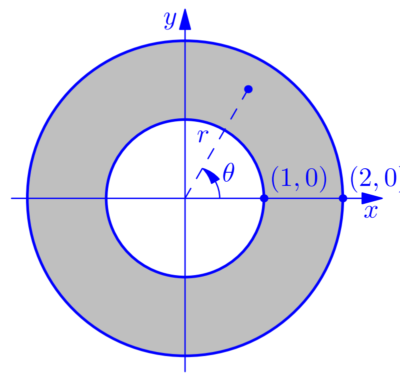
Problem 3: Evaluate the surface integral $\iint_S z^2\,dS$ where S is the portion of the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z = 2$ and $z = 4$.

Hint: in the usual polar coordinates (with $x^2 + y^2 = r^2$) the equation of the cone is $z = 2r$.

The intersection with $z = 2$ is the circle $2 = 2r \implies r = 1$.
The intersection with $z = 4$ is the circle $4 = 2r \implies r = 2$.





Use polar coordinates to parameterize S:

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta, 2r) \qquad (r,\theta) \in D = [1,2]\times[0,2\pi]$$

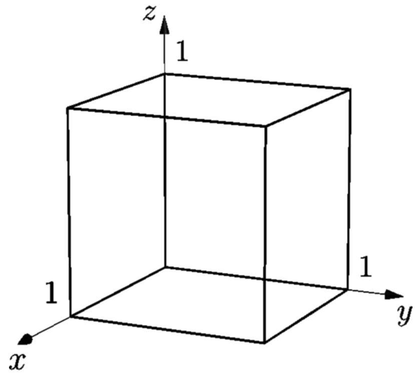
$$\implies \begin{array}{l} \mathbf{r}_r = (\cos\theta, \sin\theta, 2) \\ \mathbf{r}_\theta = (-r\sin\theta, r\cos\theta, 0) \end{array}$$

$$\implies \mathbf{r}_r \times \mathbf{r}_\theta = -2r\cos\theta\,\hat{\mathbf{i}} + 2r\sin\theta\,\hat{\mathbf{j}} + r\hat{\mathbf{k}}$$
$$\implies |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{5}r$$

So finally,

$$\begin{aligned} \iint_S z^2\,dS &= \iint_D z^2\,|\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta \\ &= \int_0^{2\pi}\int_1^2 (2r)^2\,(\sqrt{5}r\,dr\,d\theta) \\ &= 4\sqrt{5}\underbrace{\int_0^{2\pi} d\theta}_{2\pi}\ \underbrace{\int_1^2 r^3\,dr}_{\frac{1}{4}r^4\big|_1^2 = \frac{15}{4}} = \boxed{30\pi\sqrt{5}} \end{aligned}$$

/5 **Problem 4:** Evaluate the flux $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = 2x\hat{\mathbf{i}} + 3y\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ and S is the surface of the unit cube $[0,1]\times[0,1]\times[0,1]$ (all 6 faces) with outward normal.



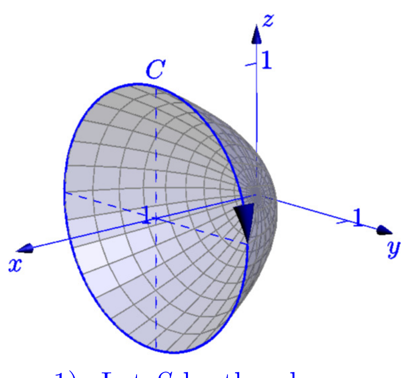
We have

$$\nabla\cdot\mathbf{F} = 2 + 3 + 2z = 5 + 2z.$$

Let D be the unit cube enclosed by S. The Divergence Theorem gives

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_D (\nabla\cdot\mathbf{F})\,dV \\
&= \iiint_D (5+2z)\,dV \\
&= 5\underbrace{\iiint_D dV}_{1} + \iiint_D 2z\,dV \\
&= 5 + \int_0^1\int_0^1\int_1 z^2\,dx\,dy\,dz \\
&= 5 + \underbrace{\int_0^1 dx}_{1}\underbrace{\int_0^1 dy}_{1}\underbrace{\int_0^1 2z\,dz}_{1} = \boxed{6}
\end{aligned}$$

/5 **Problem 5:** Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y,z) = xy\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + z^2\hat{\mathbf{k}}$ and C is the intersection of the paraboloid $x = y^2 + z^2$ and the plane $x = 1$ (with clockwise orientation when viewed from the positive x-axis).



C is the circle $y^2 + z^2 = 1$ (at $x = 1$). Let S be the planar region in the interior of this circle. The unit normal to S (consistent with the orientation of C) is $\hat{\mathbf{n}} = -\hat{\mathbf{i}}$.
We have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & x^2 & z^2 \end{vmatrix} = x\hat{\mathbf{k}}$$

Stokes' Theorem gives

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}\,dS \\
&= \iint_S \underbrace{x\hat{\mathbf{k}}\cdot(-\hat{\mathbf{i}})}_{0}\,dS = \boxed{0}
\end{aligned}$$