THOMPSON RIVERS UNIVERSITY

MATH 2670
Calculus 4 for Engineering

Instructor: Richard Taylor

MIDTERM EXAM #1 (IN-CLASS PORTION)
SOLUTIONS

15 Feb. 2019 11:30–12:20

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		5
3		5
4		7
TOTAL:		22

/5 **Problem 1:** Evaluate $\int_C (x-y+z-2)\,ds$ where C is the straight-line segment from $(0,1,1)$ to $(1,0,1)$.

Parametrize C:

$$\begin{aligned}\mathbf{r}(t) = \big(x(t), y(t), z(t)\big) &= (0,1,1) + t\big((1,0,1)-(0,1,1)\big)\\ &= (t, 1-t, 1) \quad (0 \le t \le 1).\end{aligned}$$

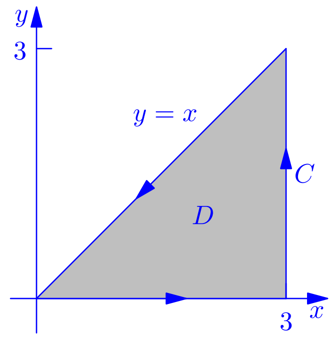
Thus

$$ds = |\mathbf{r}'(t)|\,dt = |(1,-1,0)|\,dt = \sqrt{2}\,dt$$

and so

$$\begin{aligned}\int_C (x-y+z-2)\,ds &= \int_0^1 \big(t-(1-t)+1-2\big)\sqrt{2}\,dt\\ &= \sqrt{2}\int_0^1 (2t-2)\,dt\\ &= \sqrt{2}\Big[t^2-2t\Big]_0^1 = \boxed{-2\sqrt{2}}\end{aligned}$$

/5 **Problem 2:** Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = (y^2 - x^2)\hat{\mathbf{i}} + (x^2 + y^2)\hat{\mathbf{j}}$ and C is the triangle bounded by $y = 0$, $x = 3$ and $y = x$, with counter-clockwise orientation.



We have $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}}$ where

$$P(x,y) = y^2 - x^2, \qquad Q(x,y) = x^2 + y^2.$$

Green's Theorem gives

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \iint_D (2x - 2y)\ dA \\
&= \int_0^3 \int_0^x (2x - 2y)\ dy\, dx \\
&= \int_0^3 \Big[2xy - y^2\Big]_{y=0}^{x}\ dx \\
&= \int_0^3 \left(2x^2 - x^2\right)\ dx \\
&= \int_0^3 x^2\, dx = \boxed{9}
\end{aligned}$$

/5 **Problem 3:** Evaluate $\oint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = x^2\hat{\mathbf{i}}+y^2\hat{\mathbf{j}}+z^2\hat{\mathbf{k}}$ and S is the surface of the "cylindrical can" (including the circular caps at the ends) cut from the solid cylinder $x^2+y^2=4$ between the planes $z=0$ and $z=1$.

This surface is closed, so we can use the Diverence Theorem:

$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

where D is the interior of the cylindrical can. We have

$$\nabla\cdot\mathbf{F} = 2x+2y+2z$$

so

$$\begin{aligned}\oint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_D \nabla\cdot\mathbf{F}\,dV \\ &= \underbrace{\iiint_D 2x\,dV}_{0} + \underbrace{\iiint_D 2y\,dV}_{0} + \iiint_D 2z\,dV \quad \text{(by symmetry)}\end{aligned}$$

In cylindrical coordinates we have $dV = \pi r^2\,dz = \pi(2^2)\,dz = 4\pi\,dz$ so

$$\iiint_D 2z\,dV = \int_0^1 (2z)4\pi\,dz = 4\pi\int_0^1 2z\,dz = \boxed{4\pi}$$

/7

Problem 4: Find the most general function $y(x)$ that satisfies:

(a) $y' = xy^3$.

/3

This equation is separable:

$$\begin{aligned}\frac{dy}{dx} = xy^3 &\implies \int y^{-3}dy = \int x\,dx \\ &\implies -\tfrac{1}{2}y^{-2} = \tfrac{1}{2}x^2 + C \\ &\implies \frac{1}{y^2} = A - x^2 \quad (A = -2C) \\ &\implies \boxed{y = \pm\frac{1}{\sqrt{A - x^2}} \quad (A \in \mathbb{R})}\end{aligned}$$

(b) $xy' + 3y = 2x^5$, $y(2) = 1$.

/4

This is a linear equation that we can solve using an integrating factor. First write the DE in "standard form":

$$y' + \underbrace{\frac{3}{x}}_{p(x)}y = 2x^4.$$

This gives the integrating factor

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

Multiplying the DE by μ gives

$$\underbrace{x^3y' + 3x^2y}_{\frac{d}{dx}(x^3y)} = 2x^7$$

$$\begin{aligned}&\implies x^3y = \int 2x^7\,dx = \frac{1}{4}x^8 + C \\ &\implies y = \frac{1}{4}x^5 + Cx^{-3}\end{aligned}$$

which is the general solution. We find the value of C by enforcing the "initial condition":

$$y(2) = 1 = \frac{1}{4}2^5 + C2^{-3} \implies C = -56$$

$$\implies \boxed{y = \frac{1}{4}x^5 - \frac{56}{x^3}}$$