MATH 2670 Lecture Notes

Richard Taylor, 2020

Introduction

Reading mathematics is a fairly advanced skill. Most students find that an interactive lecture format is an easier way to get acquainted with new mathematical ideas. But reading math is a skill worth learning, and this seems as good a time as any to start working on it.

The dates indicated here are the days I would have covered the material in class, and I've put them here to help you pace your work. In the right-hand margin I have indicated the relevant sections of the textbook. In some places I have suggested specific sections you should definitely read, and exercises you should attempt as you progress. I will add to this document as we go, so **check back here often**.

I've purposefully made these notes brief and to the point. You should use the textbook for supplemental information, and especially be sure to do the assigned readings. I aim to get you solving relevant problems on your own as quickly as possible. If you read these notes, do the assigned readings, and still find it difficult to get started on the problems, send me an email ASAP. I will help however I can, and your feedback will help me to make these notes better.

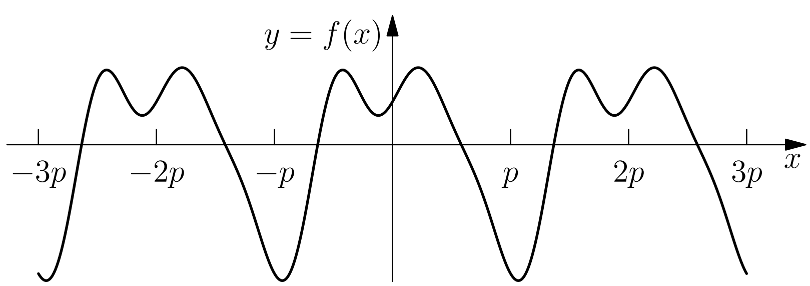
March 23

1 Fourier Series

Ch. 11

1.1 Periodic Functions

Periodic functions (like the familiar sine and cosine) play an important role in analyzing signals and systems, in many branches of engineering. The figure below shows the graph of a typical periodic function, with period equal to $2p$ (this turns out to be a convenient way to represent the period):



This function is periodic (with period $2p$) because $f(x) = f(x + 2p)$ for all x; i.e. the shape of the graph repeats itself on every interval of length $2p$.

1.2 Fourier Series

Sec. 11.2

Pretty much any periodic function you are likely to encounter will be at least piecewise continuous (i.e. its graph might have discontinuities but otherwise consists of continuous pieces). It turns out that any such function can be represented as a superposition of sines and cosines. This can be useful for a lot of reasons, not least because the sine and cosine functions are already familiar.

The *Fourier series* of a periodic function $f(x)$ with period $2p$ is given by

$$\begin{aligned} f(x) =& \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{p}\right) + a_2 \cos\left(\frac{2\pi x}{p}\right) + a_3 \cos\left(\frac{3\pi x}{p}\right) + \cdots \\ &+ b_1 \sin\left(\frac{\pi x}{p}\right) + b_2 \sin\left(\frac{2\pi x}{p}\right) + a_3 \sin\left(\frac{3\pi x}{p}\right) + \cdots \end{aligned} \tag{1}$$

or simply

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right). \tag{2}$$

Note that every sine and cosine term in eq. (2) has period $2p$ (**check this**). Consequently, the infinite sum in this formula is itself a periodic function with period $2p$.

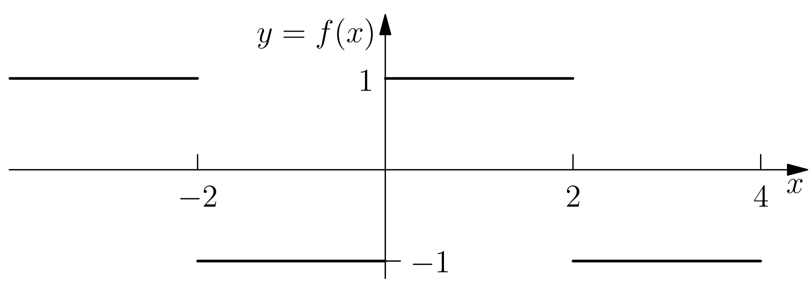
An important skill is to determine the values of the coefficients a_n, b_n for a given periodic function. These can be found using the following formulas:

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\, dx \tag{3}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\, dx. \tag{4}$$

<u>**Reading Assignment:**</u> Zill pp. 431–432. This explains where eqs. (3)–(4) come from.

Example. Calculate the Fourier series representation of the "square wave" periodic function whose graph is shown:



Solution. The period is $2p = 4$ (so $p = 2$) and on the interval $(-2, 2)$ the function is given by

$$f(x) = \begin{cases} -1 & -2 \le x < 0 \\ 1 & 0 \le x \le 2. \end{cases}$$

With $p = 2$, eq. (3) gives the cosine coefficients as

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\, dx = 0 \tag{5}$$

because the product of f (an odd function) and the cosine (an even function) is odd, so integrates to zero on a symmetric interval. Exploiting symmetry like this saves a lot of effort!

Eq. (4) gives the sine coefficients as

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)\, dx && (6)\\ &= \int_{0}^{2} \sin\left(\frac{n\pi x}{2}\right)\, dx \qquad \text{(by even symmetry)} && (7)\\ &= \left[-\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 && (8)\\ &= \frac{2}{n\pi}\big[1-\cos(n\pi)\big]. && (9) \end{aligned}$$

Note that $\cos(n\pi) = (-1)^n$ so we can write this as

$$b_n = \frac{2}{n\pi}\big[1-(-1)^n\big]. \tag{10}$$

Finally, putting this back into eq. (2) gives

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi}\big[1-(-1)^n\big]\sin\left(\frac{n\pi x}{2}\right) && (11)\\ &= \frac{4}{\pi}\sin\left(\frac{\pi x}{2}\right) + \frac{4}{3\pi}\sin\left(\frac{3\pi x}{2}\right) + \frac{4}{5\pi}\sin\left(\frac{5\pi x}{2}\right) + \cdots && (12) \end{aligned}$$

(the terms with n even are all zero). You should try plotting (via a graphing calculator or Wolfram Alpha) the graph of the sum of the first 3 terms of this series. Then try 4 terms, 5 terms, etc. and investigate what happens as you increase the number of terms. Does you graph start to resemble the square wave above? How does it differ? What does the series evaluate to at $x = 0$?

Reading Assignment: Read Zill Sec. 11.2, Examples 1 and 3.

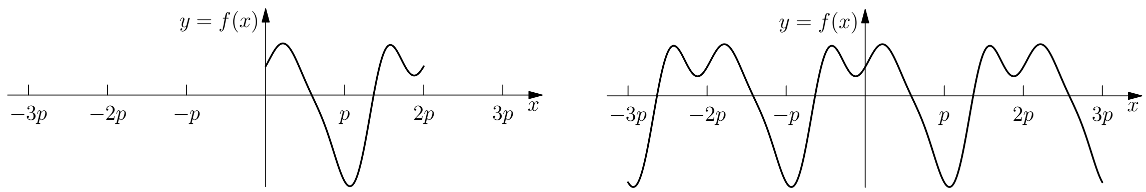
Exercises: Zill 11.2; #1, 3, 9.

March 25

Remarks

- You need to take care in using eq. (3) to evaluate a_0. When $n = 0$ the form of the integral changes, so you need to find a_0 by a separate calculation. It helps to notice that the leading term $\frac{a_0}{2}$ in (2) is just the *average value* of $f(x)$; often you can find this by inspection of the graph.

- A function need not be periodic to have a Fourier series representation. Formulas (3)–(4) for the coefficients a_n and b_n depend on the values of $f(x)$ *only* on the interval $[-p, p]$. So even if $f(x)$ is defined only on this interval, we can still calculate its Fourier series. In this case the Fourier series (which itself is defined on the whole real line) is a *periodic extension* of f.



$f(x)$ defined only on $[0, 2p]$ Periodic extension of f

- Formulas (3)–(4) for the coefficients a_n and b_n involve an integral of $f(x)$ on the interval $[-p, p]$, i.e. a whole period. There is nothing special about this interval; the formulas give the correct result as long as the integral is taken over any whole period; e.g. we could just as well integrate over $[0, 2p]$. Sometimes this can make calculations easier.

- You can think of the Fourier series (2) for $f(x)$ is a decomposition of f into a sum of sines and cosines of different frequencies. This makes most sense if we take x to be time, so that $f(x)$ is a periodic signal in time. Notice that the frequencies $n\pi/p$ are integer multiples of a fundamental frequency π/p, which corresponds to the overall period of $2p$. To musicians, these multiples of the fundamental frequency will be familiar as the *harmonics* or *upper partials* of a fundamental tone.

- The coefficients a_n, b_n together (technically, the quantity $a_n^2 + b_n^2$) have a physical interpretation as the "amount of energy in the signal" at frequency $n\pi/p$. To musicians this corresponds to the strengths of the various harmonics in a complex musical tone (the main reason a trumpet and violin sound quite different, even when playing the same note, is that the various harmonics have differing relative strengths). This connection will be important if/when you study the Fourier transform and spectral analysis. A spectrum analyzer in the lab is basically a device for calculating Fourier series coefficients.

1.3 Cosine and Sine Series

Sec. 11.3

In the square-wave example above, the function $f(x)$ had *odd* symmetry ($f(-x) = -f(x)$). Consequently, the coefficients a_n of the cosine terms in its Fourier series (2) are all zero. The Fourier series (12) for f then contains sine terms only, and we call such a series a **sine series**:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right). \tag{13}$$

In fact for **any odd function** $f(x)$, equation (3) gives $a_n = 0$ for all n. This is because the function $f(x)\cos\left(\frac{n\pi x}{p}\right)$ is odd. In eq. (3) we are integrating this function over a symmetric interval, and this integral will be zero by symmetry.

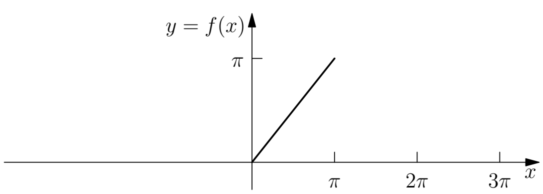
On the other hand, for **any even function** $f(x)$, equation (4) gives $b_n = 0$ for all n, because the function $f(x)\sin\left(\frac{n\pi x}{p}\right)$ is odd. The Fourier series for $f(x)$ will contain cosine terms only, and we call such a series a **cosine series**:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right). \tag{14}$$

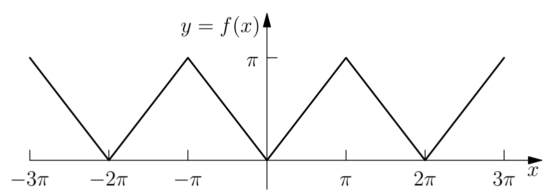
Occasionally you might encounter the following problem: $f(x)$ is given on the interval $[0, p]$, and you want to find either a cosine series or sine series representation for f. To solution to this problem is to simply extend the definition of $f(x)$ to the symmetric interval $[-p, p]$, so that the extension is either an odd function (if we want a sine series) or an even function (if we want a cosine series).

Example. Find the cosine series for the function $f(x) = x$ on the interval $[0, \pi]$.

Solution. We want a cosine series (an even function, by definition) so we first extend the definition of f to make it an even function on the symmetric interval $[-\pi, \pi]$:



original $f(x)$ on $[0, \pi]$



even (periodic) extension of f

Now we just need to calculate the Fourier series for this even extension. The coefficients b_n of the sine terms will all be zero, so we just need the coefficients a_n of the cosine terms. The period here is $2p = 2\pi$ so $p = \pi$, and eq. (3) gives:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx & (15)\\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx \qquad \text{(by symmetry, since the integrand is even)} & (16)\\
&= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \qquad \text{(integrate this by parts)} & (17)\\
&= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right]_0^{\pi} & (18)\\
&= \frac{2}{\pi}\left(\left[\frac{\pi}{n}\underbrace{\sin(n\pi)}_{0} + \frac{1}{n^2}\underbrace{\cos(n\pi)}_{(-1)^n}\right] - \left[\frac{0}{n}\underbrace{\sin(0)}_{0} + \frac{1}{n^2}\underbrace{\cos(0)}_{1}\right]\right) & (19)\\
&= \frac{2}{\pi}\frac{(-1)^n - 1}{n^2}. & (20)
\end{aligned}$$

We need to take care with the $n = 0$ term though, because the form of the integral is different in this case. Eq. (3) gives:

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx & (21)\\
&= \frac{2}{\pi}\int_{0}^{\pi} x\,dx \qquad \text{(by even symmetry)} & (22)\\
&= \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi. & (23)
\end{aligned}$$

Finally, putting everything back into eq. (2) gives:

$$\boxed{f(x) = x = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\frac{(-1)^n - 1}{n^2}\cos(nx).} \tag{24}$$

This series will agree with $f(x) = x$ on the interval $[0, \pi]$. But beyond this integral it actually represents the even periodic function in the graph above.

In general, if $f(x)$ is defined on the interval $[0, p]$ then we can use the method in the example above to derive the following formulas:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{p}\right) \qquad \textbf{(cosine series)} \tag{25}$$

where

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx. \tag{26}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{p}\right) \qquad \textbf{(sine series)} \tag{27}$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\, dx. \tag{28}$$

Reading Assignment: Zill Sec. 11.3, especially Example 3.

Exercises: Zill 11.3; #25, 27, 29, 31.

March 30

1.4 Convergence of Fourier Series

As for any other infinite series, we should be concerned about whether the Fourier series in eq. (2) converges. The theory is beyond the level of this course, but it turns out that if the original $f(x)$ is continuous then its Fourier series converges to $f(x)$ everywhere; in other words, a continuous function and its Fourier series are really the same function. However, at a point where $f(x)$ has a jump discontinuity, its Fourier series essentially interpolates between the values of its left- and right-hand limits at that point:

Theorem. *Let $f(x)$ be a periodic function and suppose that both $f(x)$ and $f'(x)$ are piecewise continuous. Then:*

- *If $f(x)$ is continuous at x then its the Fourier series converges to $f(x)$ at that point.*
- *If $f(x)$ has a jump discontinuity at x then at that point its Fourier series converges to the average*

$$\frac{f(x^+) + f(x^-)}{2}$$

  where

$$f(x^+) = \lim_{h \to 0^+} f(x+h)$$
$$f(x^-) = \lim_{h \to 0^-} f(x+h).$$

Reading Assignment: Zill Sec. 11.1, Example 3

2 Power Series Solutions of Differential Equations

Ch. 6

Our methods for solving differential equations basically consist of the following:

1. guess the right form of the solution (with some undetermined constants)
2. plug this into the DE to determine the values of the constants.

This works great if you can guess the correct form of the solution, but sometimes you can't. But recall from Calculus 2 that all functions of scientific interest have a *power series* representation

$$y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{29}$$

where a is some constant (we say the series is "centered at a"). If $f(x)$ is known then the coefficients c_n can be found using the Taylor series formula.

For our purposes here, eq. (29) is an extremely general guess as to the form for the solution of a differential equation. Faced with a DE that we can't solve any other way, our strategy is to assume the solution has the form of a power series; by plugging eq. (29) into the DE we should be able to determine the values of the coefficients c_n. The following example shows how this works in practice...

Example. Find a power series solution of the differential equation

$$y'' + y = 0.$$

We already know the solution of this problem (the general solution is $y = c_1 \cos x + c_2 \sin x$) so it makes a nice test case.

Solution. This is a bit long and involved. I've numbered the steps here to highlight the key ideas and provide a recipe you can use in other problems...

1. **Assume a power series for y, y' and y''.**

   We might as well assume $y(x)$ can be represented by a power series centered at $a = 0$ (Maclaurin series) so that

   $$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \tag{30}$$

   Differentiating term by term gives a power series for y':

   $$y' = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 x + 2c_2 x + 3c_3 x^2 + \cdots \tag{31}$$

   The $n = 0$ term is redundant (**why?**) so we can simply delete it and write

   $$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}. \tag{32}$$

   Differentiating again gives

   $$y'' = \sum_{n=1}^{\infty} n(n-1) c_n x^{n-2} \tag{33}$$

   but again the $n = 1$ term is redundant so we might as well write

   $$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \tag{34}$$

2. **Substitute into the DE to get a recurrence relation.**

   Substituting our series for y and y'' into the DE gives

   $$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0. \tag{35}$$

To get a useful equation for the coefficients c_n, we need to do two things: (1) re-index the series so each has a common factor x^n, and (2) combine the two series into one. First, let's re-index the first series by replacing n with $n+2$ to get:

$$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n + \sum_{n=0}^{\infty} c_n x^n = 0. \tag{36}$$

(You should probably write out several terms of the first series here to convince yourself that eq. (36) really is equivalent to (35)). Since the two series range over the same values of n, we can combine them into one (and factor out the common x^n):

$$\sum_{n=0}^{\infty} \underbrace{\left[(n+2)(n+1)c_{n+2} + c_n\right]}_{a_n} x^n = 0. \tag{37}$$

The left-hand side is a power series function with coefficients a_n. Since this function must be identically zero (i.e. the right-hand side) it follows that all the a_n must be zero:

$$(n+2)(n+1)c_{n+2} + c_n = 0 \qquad (n = 0, 1, 2, \ldots) \tag{38}$$

This gives a **recurrence relation** for our unknown coefficients c_n. It will help if we solve for c_{n+2} to get

$$c_{n+2} = -\frac{c_n}{(n+2)(n+1)} \qquad (n = 0, 1, 2, \ldots) \tag{39}$$

3. **Solve the recurrence relation to find the coefficients.**

Eq. (39) is really a system of infinitely many (linear) equations for infinitely many unknowns c_n. Fortunately, it isn't too hard to solve this system. With $n = 0$ and $n = 1$ eq. (39) gives

$$c_2 = -\frac{c_0}{2 \cdot 1}, \qquad c_3 = -\frac{c_1}{3 \cdot 2} \tag{40}$$

(I haven't multiplied out the denominators; this turns out to be good strategy). With $n = 2$ and $n = 3$ eq. (39) gives

$$c_4 = -\frac{c_2}{4 \cdot 3}, \qquad c_5 = -\frac{c_3}{5 \cdot 4} \tag{41}$$

but we can substitute eq. (40) into these to get

$$c_4 = \frac{c_0}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{c_0}{4!}, \qquad c_5 = \frac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{c_1}{5!}. \tag{42}$$

By the same process, with $n = 4$ and $n = 5$ eq. (39) gives

$$c_6 = -\frac{c_4}{6 \cdot 5} = -\frac{c_0}{6!}, \qquad c_7 = -\frac{c_5}{7 \cdot 6} = -\frac{c_1}{7!}. \tag{43}$$

By now the pattern should be clear, so e.g. you could immediately write a formula for, say, c_8 and c_9.

4. **Put the coefficients back into the power series.**

Putting these coefficient back into eq. (30) and simplifying gives

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \tag{44}$$

$$= c_0 + c_1 x - \frac{c_0}{2!}x^2 - \frac{c_1}{3!}x^3 + \frac{c_0}{4!}x^4 + \frac{c_1}{5!}x^5 - \cdots \tag{45}$$

$$= c_0 \underbrace{\left[1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots\right]}_{y_0(x)} + c_1 \underbrace{\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right]}_{y_1(x)}. \tag{46}$$

Note that c_0 and c_1 remain undetermined: as usual, the solution of a 2nd-order DE involved 2 constants of integration. Also, as expected because the equation is linear, the general solution is a linear combination of two independent solutions:

$$y_0(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots \tag{47}$$

$$y_1(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots . \tag{48}$$

(You might recognize these as the power series for $\cos x$ and $\sin x$.)

Reading Assignment: Sec. 6.1 Example 4.

Exercises: Sec. 6.1; #35, 37.

The End.