THOMPSON RIVERS UNIVERSITY

MATH 224
Differential Equations I

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

26 March 2010 08:30–09:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		8
2		8
3		6
4		10
5		5
TOTAL:		37

/8

Problem 1: Find the general solution of the following differential equation:

$$2y''' - 4y'' - 2y' + 4y = 0.$$

$$y = e^{rx} \implies 2r^3 - 4r^2 - 2r + 4 = 0$$

By inspection, $r = 1$ is a root; by polynomial division, we get

$$\frac{2r^3 - 4r^2 - 2r + 4}{r - 1} = 2r^2 - 2r - 4 = 2(r^2 - r - 2) = 2(r - 2)(r + 1)$$

so the other roots are $r = 2$, $r = -1$.

We have three distince real roots of the characteristics equation, so the general solution is

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x} + c_3 e^{-x}}$$

/8 **Problem 2:** Consider the following fourth-order differential equation

$$y^{(4)} + 7y'' + 6y = 0.$$

(a) Find the general solution of this equation.

/5

$$\begin{aligned} y = e^{rx} &\implies r^4 + 7r^2 + 6 = 0 \\ &\implies (r^2+6)(r^2+1) = 0 \\ &\implies r = \pm i,\ \pm\sqrt{6}i \end{aligned}$$

We have four roots (in complex conjugate pairs) so the general solution is

$$\boxed{y(x) = c_1 \cos x + c_2 \sin x + c_3 \cos \sqrt{6}x + c_4 \sin \sqrt{6}x}$$

(b) Find the solution that satisfies the initial conditions

/3

$$y(0) = 6, \quad y'(0) = 0, \quad y''(0) = -11, \quad y'''(0) = 0.$$

$$y(0) = 6 = c_1 + c_3 \qquad (1)$$

$$y'(0) = 0 = c_2 + \sqrt{6}c_4 \qquad (2)$$

$$y''(0) = -11 = -c_1 - 6c_3 \qquad (3)$$

$$y'''(0) = 0 = -c_2 - 6^{3/2}c_4 \qquad (4)$$

Equations (2) and (4) together give $c_2 = c_4 = 0$.

Adding equations (1) and (3) gives $-5c_3 = -5 \implies c_3 = 1 \implies c_1 = 5$.

The solution is therefore

$$\boxed{y(x) = 5\cos x + \cos\sqrt{6}x}$$

/6

Problem 3: The functions

$$y_1(t) = 1, \quad y_2(t) = t, \quad y_3(t) = e^{-t}, \quad y_4(t) = te^{-t}$$

are all solutions of the differential equation

$$y^{(4)} + 2y''' + y'' = 0.$$

Prove that $\{y_1, y_2, y_3, y_4\}$ is a fundamental set of solutions. What can you conclude about the general solution?

Consider the Wronskian

$$W(t) = \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & e^{-t} - te^{-t} \\ 0 & 0 & e^{-t} & -2e^{-t} + te^{-t} \\ 0 & 0 & -e^{-t} & 3e^{-t} - te^{-t} \end{vmatrix}$$

Cofactor expansion along the first column gives

$$\begin{aligned} W(t) &= (1)(1) \begin{vmatrix} e^{-t} & -2e^{-t} + te^{-t} \\ -e^{-t} & 3e^{-t} - te^{-t} \end{vmatrix} \\ &= (e^{-t})(3e^{-t} - te^{-t}) - (-e^{-t})(-2e^{-t} + te^{-t}) \\ &= 3e^{-2t} - te^{-2t} - 2e^{-2t} + te^{-2t} \\ &= e^{-2t} \end{aligned}$$

Since $W(t) = e^{-2t} \neq 0 \ \forall t$, the solutions are linearly independent and therefore form a fundamental set of solutions.

The general solution of the differential equation is a linear combination

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 \\ &= \boxed{c_1 + c_2 t + c_3 e^{-t} + c_4 te^{-t}} \end{aligned}$$

/10

Problem 4: Consider the following system of differential equations:

$$\begin{cases} x' = 2x - y \\ y' = 3x - 2y. \end{cases}$$

(a) Find the general solution of this system.

/6

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$$

Calculate eigenvalues of A:

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 3 = \lambda^2 - 1 \implies \lambda = \pm 1$$

Calculate eigenvectors:
$\lambda_1 = 1$:

$$(A - (1)I)\mathbf{v}_1 = \mathbf{0} \implies \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_1 = (1,1).$$

$\lambda_2 = -1$:

$$(A - (-1)I)\mathbf{v}_2 = \mathbf{0} \implies \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \mathbf{v}_2 = (1,3).$$

We have distinct real eigenvalues (hence linearly independent eigenvectors) so the general solution is the linear combination

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} \implies \boxed{\begin{cases} x(t) = c_1 e^t + c_2 e^{-t} \\ y(t) = c_1 e^t + 3c_2 e^{-t} \end{cases}}$$

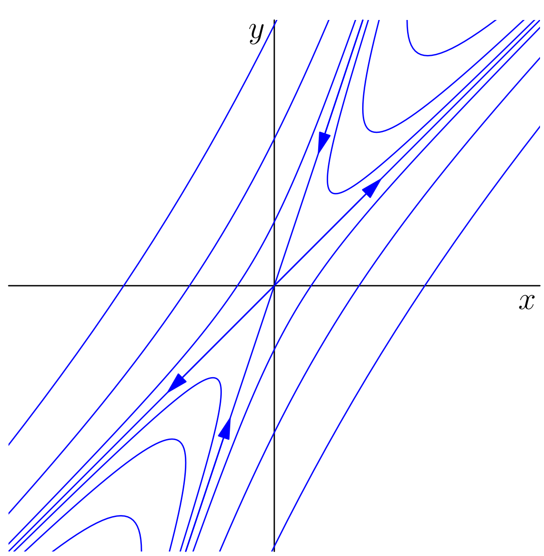
(b) Classify the equilibrium $(0,0)$ as to type and stability.

/2

$(0,0)$ is a saddle point (unstable).

(c) Accurately sketch the phase portrait.

/2



/5

Problem 5: For the system of differential equations

$$\begin{cases} x' = -x \\ y' = 4x - y \\ z' = 3x + 6y - 2z \end{cases}$$

is the equilibrium at $x = y = z = 0$ unstable? stable? asympotically stable?

$$\mathbf{x}' = A\mathbf{x}, \quad A = \begin{bmatrix} -1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 6 & -2 \end{bmatrix}$$

Calculate eigenvalues of A:

$$0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 0 \\ 4 & -1-\lambda & 0 \\ 3 & 6 & -2-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda)(-2-\lambda)$$

$$\implies \lambda = -1, -1, -2$$

We have a repeated eigenvalue; depending on geometric multitiplicity the general solution is

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{-2t} + c_2\mathbf{v}_2 e^{-t} + c_3\mathbf{v}_3 e^{-t}$$

or

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{-2t} + c_2\mathbf{v}_2 e^{-t} + c_3(t\mathbf{v}_2 + \mathbf{b})e^{-t}.$$

In either case we have

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$$

so the equilibrium $\mathbf{x} = \mathbf{0}$ is asymptotically stable.