THOMPSON RIVERS UNIVERSITY

MATH 224
Differential Equations I

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MIDTERM EXAM #1
SOLUTIONS

26 February 2010 08:30–09:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 5 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		8
2		6
3		6
4		8
5		6
TOTAL:		34

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Problem 1: Consider the autonomous differential equation

$$\frac{dy}{dt} = y^2(y^2 - 1).$$

(a) Determine the equilibrium solutions of this equation.

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$$\frac{dy}{dt} = y^2(y^2 - 1) = 0 \implies \boxed{y = 0, \pm 1}$$

(b) Draw the one-dimensional phase portrait and determine the stability (asymtotically stable, unstable or semi-stable) of each equilibrium.

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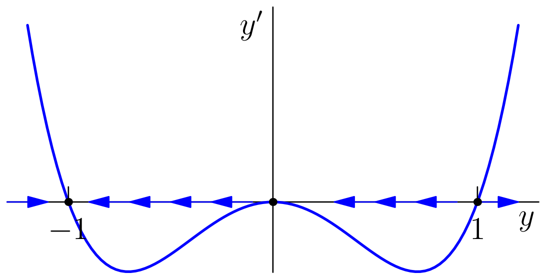

$y = 1$ is unstable
$y = 0$ is semi-stable
$y = -1$ is asymptotically stable

(c) Sketch the graph of several solutions in the ty-plane.

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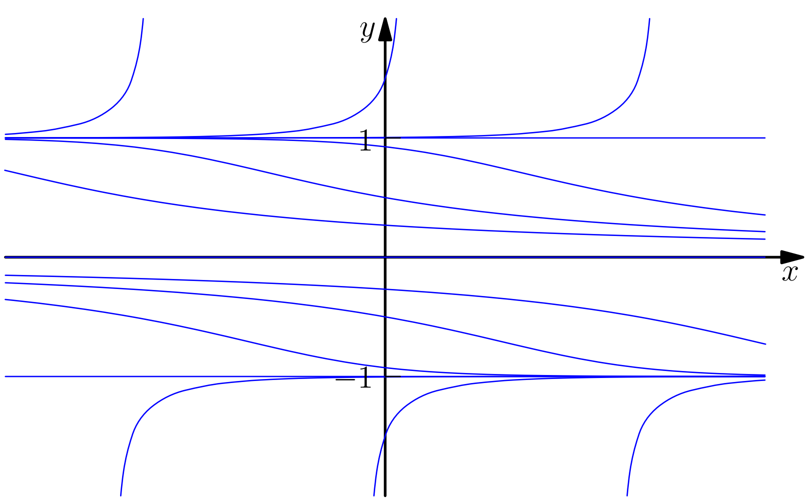

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Problem 2: Solve the initial value problem

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}, \quad y(0) = -2.$$

Separable equation:

$$\frac{dy}{dx} = \frac{2x}{y(1+x^2)} \implies \int y\,dy = \int \frac{2x}{1+x^2}\,dx$$
$$\implies \frac{1}{2}y^2 = \ln(1+x^2) + C$$
$$\implies y = \pm\sqrt{2\ln(1+x^2) + C}$$

Impose initial conditions:

$$y(0) = -2 = \pm\sqrt{2\ln(1) + C} \implies 4 = C$$

$$\implies \boxed{y = -\sqrt{2\ln(1+x^2) + 4}}$$

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Problem 3: Solve the differential equation

$$x^3 y' + 4x^2 y = e^{-x}.$$

Linear equation with integrating factor

$$\mu(x) = e^{\int 4/x\,dx} = e^{4\ln x} = e^{\ln x^4} = x^4.$$

Multiplying both sides by μ gives

$$\underbrace{x^4 y' + 4x^3 y}_{\frac{d}{dx}(x^4 y)} = x e^{-x} \implies x^4 y = \int x e^{-x}\,dx$$

$$\implies y = \frac{1}{x^4}\int x e^{-x}\,dx = \frac{1}{x^4}\left[-(x+1)e^{-x} + C\right]$$

$$\implies \boxed{y = \frac{C}{x^4} - \frac{x+1}{x^4}e^{-x}}$$

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Problem 4: Consider the differential equation

$$y'' + 9y = f(x).$$

(a) Find the general solution in the case where $f(x) = xe^{3x}$.

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Homogeneous case:

$$y'' + 9y = 0 \implies y_h = c_1 \cos 3x + c_2 \sin 3x$$

Use undetermined coefficients to find a particular solution:

$$\begin{aligned} & y_p = (A + Bx)e^{3x} \\ \implies & y_p' = 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} \\ \implies & y_p'' = 9Ae^{3x} + 3Be^{3x} + 3Be^{3x} + 9Bxe^{3x} \end{aligned}$$

$$\implies \underbrace{(9A + 6B + 9Bx)e^{3x}}_{y_p''} + 9\underbrace{(A + Bx)e^{3x}}_{y_p} = xe^{3x}$$

Match coefficients:

$$\begin{cases} 18A + 6B = 0 \\ 9B + 9B = 1 \end{cases} \implies B = \frac{1}{18},\ A = -\frac{1}{54}$$

So the general solution is:

$$\boxed{y = c_1 \cos 3x + c_2 \sin 3x + \left(\frac{x}{18} - \frac{1}{54}\right) e^{3x}}$$

(b) For the case $f(x) = x \cos 3x$, determine a suitable form for the particular solution if the method of undetermined coefficients is to be used. *Do not attempt to determine the coefficients, and do not attempt to find the general solution.*

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$$y_p = x\left[(A + Bx)\cos 3x + (C + Dx)\sin 3x\right]$$

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Problem 5: Consider the differential equation

$$y'' + 2y' - 3y = g(x).$$

Use variation of parameters to find the general solution of this equation. Express your answer in terms of an appropriate integral involving $g(x)$.

Solve the corresponding homogeneous equation (i.e. with $g(x) = 0$):

$$y = e^{rx} \implies r^2 + 2r - 3 = (r+3)(r-1) = 0 \implies r = 1,\ -3$$

So two homogeneous solutions are

$$y_1 = e^x \qquad y_2 = e^{-3x}.$$

Now use variation of parameters: seek a particular solution of the form

$$y_p = u_1 y_1 + u_2 y_2$$

This yields the following system of equations:

$$\begin{cases} u_1' e^x + u_2' e^{-3x} = 0 \\ u_1' e^x - 3u_2' e^{-3x} = f(x) \end{cases} \implies u_1' = \frac{1}{4} e^{-x} f(x),\ u_2' = -\frac{1}{4} e^{3x} f(x)$$

Integrating these gives

$$\begin{aligned} y_p &= e^x \int_0^x \frac{1}{4} e^{-s} f(s)\, ds + e^{-3x} \int_0^x \left(-\frac{1}{4}\right) e^{3s} f(s)\, ds \\ &= \frac{1}{4} \int_0^x \left[e^x e^{-s} - e^{-3x} e^{3s}\right] f(s)\, ds \\ &= \frac{1}{4} \int_0^x \left[e^{x-s} - e^{-3(x-s)}\right] f(s)\, ds \end{aligned}$$

so the general solution is

$$\boxed{y = c_1 e^x + c_2 e^{-3x} + \int_0^x G(x-s) f(s)\, ds}$$

where

$$G(t) = \frac{1}{4}\left(e^t - e^{-3t}\right).$$