Name: _____ Student #: _____

THOMPSON RIVERS UNIVERSITY

MATH 224
Differential Equations I

Instructor: Richard Taylor

FINAL EXAM

16 April 2010 09:00–12:00

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 8 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		6
3		7
4		6
5		10
6		5
7		6
8		11
TOTAL:		63

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Problem 1: Solve each of the following:

/4 (a) $t\dfrac{dx}{dt} = 4x + t^4; \quad x(1) = 5$

/4 (b) $\dfrac{dy}{dx} = \dfrac{2xy + 3}{1 - x^2}$

/4 (c) $\dfrac{dy}{dt} = 2\sqrt{y+1}\cos t; \quad y(\pi) = 0$

/6 **Problem 2:** Consider the autonomous first-order differential equation

$$\frac{dy}{dt} = \cos 2y.$$

(a) Find the equilibrium solutions of this equation. Classify each equilibrium as either unstable, asymptotically stable, or semi-stable. Sketch the one-dimensional phase portrait.

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(b) Sketch the equilibrium solutions in the ty-plane. These solutions divide the plane into several regions. Sketch at least one solution in each of these regions.

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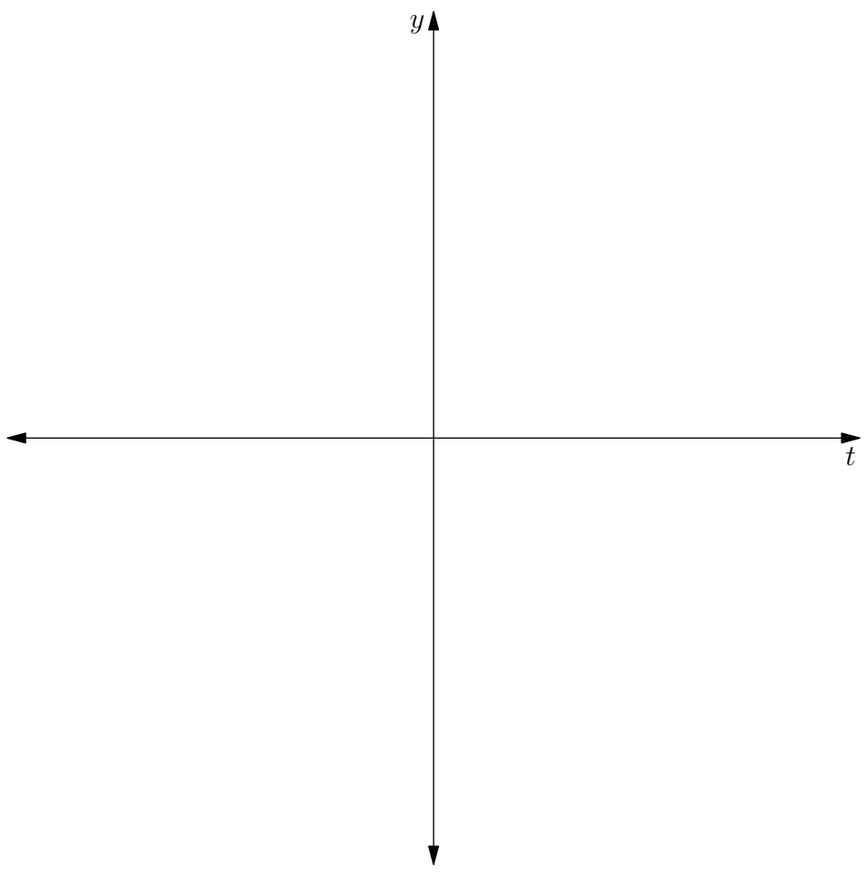

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Problem 3: A 600-gallon tank is initially filled with 300 gallons of pure water. Salt solution containing 1.5 lb of salt per gallon flows into the tank at a rate of 3 gal/min. The tank also has a drain, through which the (well-mixed) contents of the tank flow out at a fixed rate of 1 gal/min.

(a) Let $x(t)$ represent the amount of sale (in lb) in the tank at time t (minutes). Show that $x(t)$ must satisfy the following differential equation:

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$$\frac{dx}{dt} + \frac{x}{300 + 2t} = 4.5$$

(b) What will be the quantity of salt in the tank at the precise moment that the solution fills the tank to capacity (600 gal)?

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Problem 4: Find the general solution $y(x)$ of the following:

$$y'' + 2y' + 2y = 2 + \cos 2x$$

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Problem 5: Consider the non-homogeneous second-order differential equation

$$x^2y'' + 3xy' - 3y = \frac{1}{x} \qquad (x > 0). \tag{1}$$

(a) Verify (by any method) that $y_1 = x$ and $y_2 = x^{-3}$ are both solutions of the corresponding homogeneous equation, and prove that they are linearly independent for $x > 0$.

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(b) Use variation of parameters to find a particular solution of equation (1).

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(c) Find the general solution of equation (1). How do you know there are no other solutions?

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Problem 6: Find the general solution of

$$y^{(4)} + 3y''' - 4y' = 0.$$

/6

Problem 7: Solve the following system of linear differential equations.

$$\mathbf{x}'(t) = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t); \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

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Problem 8: Van der Pol's equation

$$x'' + \varepsilon(x^2 - 1)x' + x = 0$$

is a model for oscillation with nonlinear damping. (ε is a constant.)

/2 (a) Write this equation as an equivalent system of first-order equations.

/1 (b) Show that the origin is the only equilibrium solution of this system.

/3 (c) Classify the equilibrium as to type and stability when $\varepsilon = 1$. Sketch the phase portrait.

/3 (d) Classify the equilibrium as to type and stability when $\varepsilon = 0$. Sketch the phase portrait.

/2 (e) For what value(s) of ε is the equilibrium a stable spiral?