THOMPSON RIVERS UNIVERSITY

MATH 2240
Differential Equations 1

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MIDTERM EXAM #2
SOLUTIONS

26 March 2015 13:00–14:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		9
2		5
3		5
4		7
5		7
TOTAL:		33

/9

Problem 1: Solve the following:

/3 (a) $y'' - 4y' + 5y = 0$

$$y = e^{rx} \implies r^2 - 4r + 5 = 0 \implies r = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

$$\therefore \boxed{y(x) = e^{2x}(c_1 \cos x + c_2 \sin x); \quad c_1,\, c_2 \in \mathbb{R}}$$

/3 (b) $y'' + 8y' + 16y = 0$

$$y = e^{rx} \implies 0 = r^2 + 8r + 16 = (r+4)^2 \implies r = -4 \text{ (repeated root)}$$

$$\therefore \boxed{y(x) = c_1 e^{-4x} + c_2 x e^{-4x}; \quad c_1,\, c_2 \in \mathbb{R}}$$

/3 (c) $y''' - y' = 0$

$$y = e^{rx} \implies 0 = r^3 - r = r(r^2 - 1) \implies r = 0,\, \pm 1$$

$$\therefore \boxed{y(x) = c_1 + c_2 e^{x} + c_3 e^{-x}; \quad c_1,\, c_2,\, c_3 \in \mathbb{R}}$$

/5

Problem 2: Find the general solution of $y'' - 16y = 2e^{4x}$.

Homogeneous problem: $y'' - 16y = 0$

$$y = e^{rx} \implies r^2 - 16 = 0 \implies r = \pm 4 \implies \begin{matrix} y_1 = e^{4x} \\ y_2 = e^{-4x} \end{matrix}$$

To find a particular solution use undetermined coefficients:

$$\begin{aligned} &y = Axe^{4x} \quad \text{(since } e^{4x} \text{ satisfies the homogeneous equation)} \\ &y' = Ae^{4x} + 4Axe^{4x} \\ &y'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} \end{aligned}$$

$$\begin{aligned} &\implies y'' - 16y = (8Ae^{4x} + 16Axe^{4x}) - 16(Axe^{4x}) = 2e^{4x} \\ &\implies 8A = 2 \implies A = \frac{1}{4} \\ &\implies y_p = \frac{1}{4}xe^{4x} \end{aligned}$$

So the general solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 + y_p \\ &= \boxed{c_1e^{4x} + c_2e^{-4x} + \frac{1}{4}xe^{4x}; \quad c_1,\, c_2 \in \mathbb{R}} \end{aligned}$$

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Problem 3: Find the general solution of $y'' + y = \tan x$.

Homogeneous problem: $y'' + y = 0$

$$y = e^{rx} \implies r^2 + 1 = 0 \implies r = \pm i \implies \begin{matrix} y_1 = \cos x \\ y_2 = \sin x \end{matrix}$$

To find a particular solution use variation of parameters: seek a solution $y = u_1y_1 + u_2y_2$

$$\implies \begin{cases} u_1' \cos x + u_2' \sin x = 0 \\ -u_1' \sin x + u_2' \cos x = \tan x \end{cases} \implies \begin{matrix} u_1' = -\sin x \tan x \\ u_2' = \cos x \tan x = \sin x \end{matrix}$$

so

$$u_1(x) = -\int_0^x \sin s \tan s \, ds \qquad \text{and} \qquad u_2(x) = \int \sin x \, dx = -\cos x$$

$$\implies y_p = -\cos x \int_0^x \sin s \tan s \, ds - \cos x \sin x$$

and the general solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 + y_p \\ &= \boxed{c_1 \cos x + c_2 \sin x - \cos x \int_0^x \sin s \tan s \, ds - \cos x \sin x; \quad c_1,\, c_2 \in \mathbb{R}} \\ &= c_1 \cos x + c_2 \sin x - \cos x \ln\left(\frac{\sin\frac{x}{2} + \cos\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right); \quad c_1,\, c_2 \in \mathbb{R} \end{aligned}$$

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Problem 4: Consider the linear system $\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 4x + 3y \end{cases}$

(a) Sketch an accurate phase portrait for the system.

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$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\begin{matrix} \Delta = \det(A) = -5 \\ \tau = \text{trace}(A) = 4 \end{matrix} \implies (0,0) \text{ is a saddle point.}$$

Eigenvalues of A:

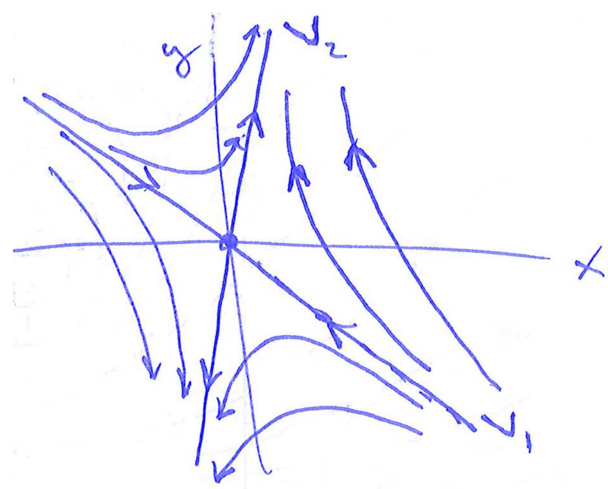
$$0 = \det A - \lambda I = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$$

$$\implies \lambda = -1,\, 5$$

Eigenvectors: $(A - \lambda I)\mathbf{v} = \mathbf{0}\ldots$

$$\lambda = -1: \quad \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 4 & 4 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies v_2 = -v_1 \implies \mathbf{v} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix},\ t \in \mathbb{R}$$

$$\lambda = 5: \quad \left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies v_2 = 2v_1 \implies \mathbf{v} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix},\ t \in \mathbb{R}$$



(b) Find the solution with initial condition $x(0) = 1$, $y(0) = 5$.

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From above we have

$$\mathbf{x}(t) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t}; \quad c_1, c_2 \in \mathbb{R}$$

Thus

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \implies \begin{matrix} c_1 = -1 \\ c_2 = 2 \end{matrix}$$

and so

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{5t} \implies \boxed{\begin{matrix} x(t) = -e^{-t} + 2e^{5t} \\ y(t) = e^{-t} + 4e^{5t} \end{matrix}}$$

/7 **Problem 5:** Consider the nonlinear system $\begin{cases} \dfrac{dx}{dt} = 1 - xy \\ \dfrac{dy}{dt} = xy - y \end{cases}$

(a) Find all the equilibria of this system and classify them according to stability.

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Let $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ then $\mathbf{x}' = \begin{bmatrix} 1 - xy \\ xy - y \end{bmatrix} = f(\mathbf{x})$. To find the equilibria:

$f(\mathbf{x}) = \mathbf{0} \implies \begin{cases} 1 - xy = 0 \\ (x-1)y = 0 \end{cases} \implies y = 0$ (contradicting the 1st equation) or $x = 1 (\implies y = 1)$.

So the only equilibrium is at $(1, 1)$. Now we linearize the system about $(1, 1)$:

$$Df(\mathbf{x}) = \begin{bmatrix} -y & -x \\ y & x - 1 \end{bmatrix} \implies Df(1,1) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

This gives $\tau = -1$, $\Delta = 1$. So $(1, 1)$ is a **stable spiral**, since $\tau < 0$ with $\Delta > \tau^2/4$.

(b) Sketch the phase portrait of the system.

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Linearization is not quite enough for an accurate phase portrait. It helps to notice that

$$y = 0 \implies \frac{dy}{dt} = 0$$

so one solution lies on the line $y = 0$. Also, it helps to sketch the null-clines which are given by

$$x' = 0 \implies y = \tfrac{1}{x}$$
$$y' = 0 \implies x = 1 \text{ or } y = 0$$