THOMPSON RIVERS UNIVERSITY

MATH 2240
Differential Equations I

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

28 March 2012 08:30–09:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 4 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		10
2		10
3		5
4		6
TOTAL:		31

/10 **Problem 1:** Evaluate the matrix exponential e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ using any method of your choice.

First find a fundamental matrix of solutions of $\mathbf{x}' = A\mathbf{x}$. For this we need eigenvalues of A:

$$0 = \det(A - rI) = \begin{vmatrix} -r & 1 \\ -1 & -r \end{vmatrix} = r^2 + 1 \implies r = \pm i$$

and the corresponding eigenvectors $\mathbf{v} = (v_1, v_2)$:

$$(A - (i)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \end{bmatrix} \implies iv_1 = v_2 \implies \mathbf{v} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which yield the following linearly independent solutions of $\mathbf{x}' = A\mathbf{x}$:

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

$$\mathbf{x}_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

This gives the fundamental matrix

$$\Phi(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

such that

$$\mathbf{x}(t) = \underbrace{\Phi(t)\Phi(0)^{-1}}_{e^{At}} \mathbf{x}(0).$$

Thus

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

/10 **Problem 2:** Solve the following initial value problem: $\begin{cases} x' = x - y & x(0) = 4 \\ y' = 4x - 3y & y(0) = 5 \end{cases}.$

We have $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$. Eigenvalues r of A are given by

$$0 = \det(A - rI) = \begin{vmatrix} 1-r & -1 \\ 4 & -3-r \end{vmatrix} = (1-r)(-3-r) + 4 = r^2 + 2r + 1 = (r+1)^2 \implies r = -1.$$

The corresponding eigenvector $\mathbf{v} = (v_1, v_2)$ is given by

$$(A - (-1)I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies 2v_1 = v_2 \implies \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This gives one solution

$$\mathbf{x}_1(t) = \mathbf{v}e^{-t}.$$

We can find a second, linearly independent solution of the form

$$\mathbf{x}_2(t) = (\mathbf{v}t + \mathbf{b})e^{-t}$$

where $\mathbf{b}$ is a "generalized eigenvector" satisfying

$$(A - (-1)I)\mathbf{b} = \mathbf{v} \implies \begin{bmatrix} 2 & -1 & 1 \\ 4 & -2 & 2 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies 2b_1 - 1 = b_2 \implies \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus the general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{-t}.$$

Now impose the initial conditions:

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \implies c = 1,\ c_2 = 3.$$

Therefore

$$\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t} + 3 \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) e^{-t}$$

or in terms of the component functions $x(t)$, $y(t)$:

$$\boxed{\begin{aligned} x(t) &= (3t+4)e^{-t} \\ y(t) &= (6t+5)e^{-t} \end{aligned}}$$

/5

Problem 3: Let A by an $n \times n$ constant matrix, and consider the system of differential equations

$$t\mathbf{x}'(t) = A\mathbf{x}(t), \quad t > 0.$$

Show that if this system has a nontrivial solution of the form $\mathbf{x}(t) = t^r\mathbf{v}$, then r is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

For $\mathbf{x}(t) = t^r\mathbf{v}$ we have

$$t\mathbf{x}' = t(rt^{r-1}\mathbf{v}) = rt^r\mathbf{v}.$$

Assume the system has a solution of this form. Substitution into the DE gives

$$\begin{aligned} rt^r\mathbf{v} = At^r\mathbf{v} &\implies (A - rI)t^r\mathbf{v} = \mathbf{0} \\ &\implies (A - rI)\mathbf{v} = \mathbf{0}. \end{aligned}$$

Thus r is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$.

/6

Problem 4: For the linear system $\begin{cases} x' = -x - 2y \\ y' = 8x - y \end{cases}$ classify the stability type of the equilibrium solution $(x, y) = (0, 0)$ and sketch the phase portrait.

We have $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} -1 & -2 \\ 8 & -1 \end{bmatrix}$. This gives

$$\begin{aligned} \Delta &= \det(A) = (-1)(-1) - (-2)(8) = 17 \\ \tau &= \operatorname{trace}(A) = -1 + -1 = -2 \end{aligned}$$

Since $4\Delta > \tau^2$ the origin is a spiral; and since $\tau < 0$ it must be a *stable spiral.*