THOMPSON RIVERS UNIVERSITY

MATH 2240
Differential Equations 1

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MIDTERM EXAM #1
SOLUTIONS

5 Feb 2015 13:00–14:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		6
2		10
3		5
4		6
TOTAL:		27

/6

Problem 1: Consider the differential equation

$$\frac{dx}{dt} = e^{-x}(x^2 - 4x).$$

(a) Find the equilibrium (constant) solutions and classify them according to stability.

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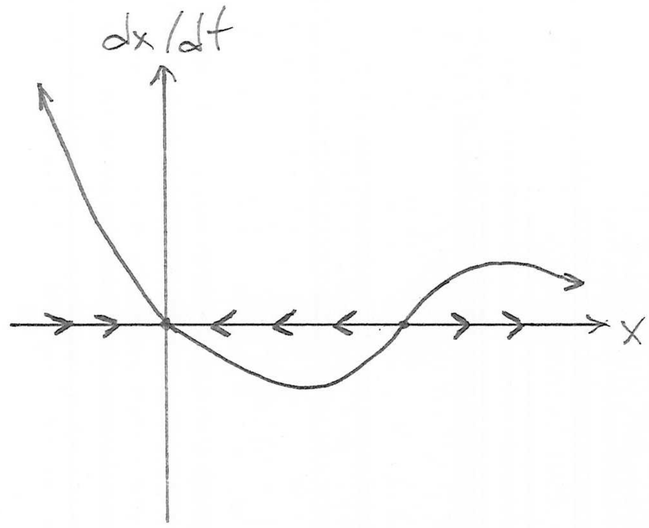
$\frac{dx}{dt} = 0 \Rightarrow x = 0, 4$ (equilibrium solutions)



$x=0$ is asymptotically stable

$x=4$ is unstable

(b) Sketch qualitative solutions in the (t, x)-plane.

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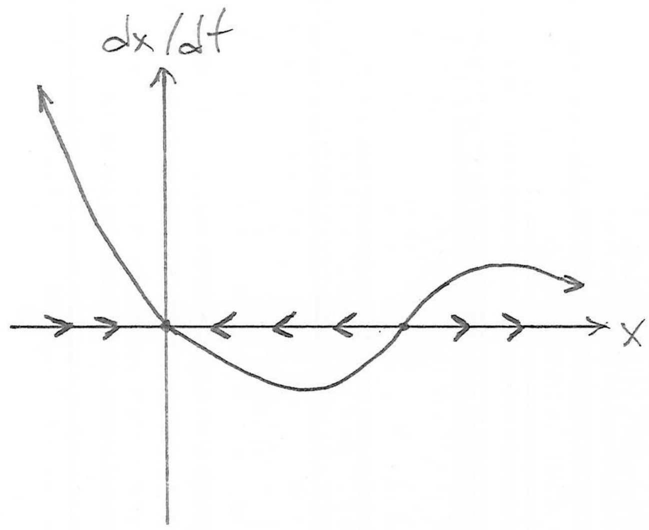
$\frac{dx}{dt} = 0 \Rightarrow x = 0, 4$ (equilibrium solutions)



$x=0$ is asymptotically stable

$x=4$ is unstable

/10 **Problem 2:** Solve the following:

/5 (a) $\dfrac{dy}{dx} = \dfrac{xy^3}{\sqrt{1+x^2}}; \qquad y(0) = -1.$

This equation is separable:

$$\frac{dy}{y^3} = \frac{x\,dx}{\sqrt{1+x^2}} \implies -\frac{1}{2y^2} = \sqrt{1+x^2} + C$$

Imposing the "initial condition" gives:

$$-\frac{1}{2(1)^2} = \sqrt{1+0^2} + C \implies C = -\frac{3}{2}$$
$$\implies -\frac{1}{2y^2} = \sqrt{1+x^2} - \frac{3}{2}$$

Solving algebraically for y gives:

$$y^2 = \frac{1}{3 - 2\sqrt{1+x^2}} \implies y = \pm\sqrt{\frac{1}{3-2\sqrt{1+x^2}}}$$

The positive root is inconsistent with the initial condition, so

$$\boxed{y = -\sqrt{\frac{1}{3-2\sqrt{1+x^2}}}}$$

/5 (b) $2x\dfrac{dy}{dx} - y = x + 1; \qquad y(2) = 4$

The DE is linear; dividing through by $2x$ puts it in "standard form":

$$\frac{dy}{dx} \underbrace{-\frac{1}{2x}}_{p(x)} y = \frac{1}{2} + \frac{1}{2x}$$

which gives the integrating factor

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int -\frac{1}{2x}\,dx} = e^{-\frac{1}{2}\ln x} = e^{\ln x^{-1/2}} = x^{-1/2}.$$

Multiplying the DE by μ gives

$$\underbrace{x^{-1/2}\frac{dy}{dx} - \frac{1}{2}x^{-3/2}y}_{\frac{d}{dx}(x^{-1/2}y)} = \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}$$

$$\implies x^{-1/2}y = \int \left(\frac{1}{2}x^{-1/2} + \frac{1}{2}x^{-3/2}\right)dx$$
$$= x^{1/2} - x^{-1/2} + C$$
$$\implies y = x - 1 + C\sqrt{x}$$

The "initial conditions" require

$$y(2) = 4 = (2) - 1 + C\sqrt{2} \implies C = \frac{3}{\sqrt{2}}$$

$$\implies \boxed{y = x - 1 + 3\sqrt{\frac{x}{2}}}$$

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Problem 3: Find the general solution of the differential equation

$$2xy^2 + 4 = 2(3 - x^2 y)\frac{dy}{dx}.$$

Suspecting the DE is exact, we re-arrange it in "standard form":

$$\underbrace{(2xy^2 + 4)}_{M}\, dx + \underbrace{2(x^2 y - 3)}_{N}\, dy = 0$$

We have

$$\frac{\partial M}{\partial y} = 4xy \qquad \frac{\partial N}{\partial x} = 4xy$$

which agree, so the DE is indeed exact. So we seek a solution of the form

$$f(x, y) = C$$

which implies

$$\underbrace{\frac{\partial f}{\partial x}}_{M}\, dx + \underbrace{\frac{\partial f}{\partial y}}_{N}\, dy = 0$$

and thus

$$\begin{cases} \dfrac{\partial f}{\partial x} = M = 2xy^2 + 4 & \implies f(x, y) = x^2y^2 + 4x + K(y) \\ \dfrac{\partial f}{\partial y} = N = 2x^2 y - 6 & = 2x^2 y + K'(y) \end{cases}$$

so that

$$K'(y) = -6 \implies K(y) = -6y$$

and the (implicit) solution of the given DE is

$$\boxed{x^2y^2 + 4x - 6y = C \in \mathbb{R}}$$

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Problem 4: At SeaWorld (in San Diego) two orca whales live in a tank containing 24 million litres (L) of water. A staff member measures the concentration of dissolved oxygen in the tank and finds that it is only 10 mg/L — not enough to support healthy whales. Fresh water containing dissolved oxygen at 20 mg/L is then pumped into the tank at a rate of 1000 L/min. Well-mixed water is drained from the tank at the same rate. After how many minutes will the concentration of dissolved oxygen in the tank reach 15 mg/L?

Let $x(t)$ be the amoung [mg] of dissolved oxygen in the tank after t minutes. Then

$$\begin{aligned}\frac{dx}{dt} &= \text{“rate in”} \;-\; \text{“rate out”}\\ &= 1000\,[\mathrm{L/min}]\cdot 20\,[\mathrm{mg/L}] - 1000\,[\mathrm{L/min}]\cdot\frac{x\,[\mathrm{mg}]}{24\times10^6\,[\mathrm{L}]}\\ &= 2\times10^4 - \frac{x}{24\times10^3}\end{aligned}$$

This DE is separable (also linear):

$$\frac{dx}{2\times10^4 - \dfrac{x}{24\times10^3}} = dt \implies \frac{dx}{(2\times10^4)(24\times10^3)-x} = \frac{dt}{24\times10^3}.$$

Integrating gives the general solution:

$$\begin{aligned}-\ln(48\times10^7 - x) = \frac{t}{24\times10^3} + C &\implies 48\times10^7 - x = e^{-t/(24\times10^3)+C} = Ae^{-t/(24\times10^3)}\\ &\implies x(t) = 48\times10^7 - Ae^{-t/(24\times10^3)}.\end{aligned}$$

The initial conditions require

$$x(0) = 10\,[\mathrm{mg/L}]\cdot 24\times10^6\,[\mathrm{L}] \quad \begin{aligned}&= 24\times10^7\,[\mathrm{mg}]\\ &= 48\times10^7 - A\end{aligned} \quad\implies A = 24\times10^7$$

so the solution of the DE is

$$x(t) = \left(48 - 24e^{-t/(24\times10^3)}\right)\times10^7.$$

The concentration reaches 15 mg/L when

$$x = 15\,[\mathrm{mg/L}]\cdot 24\times10^6\,[\mathrm{L}] = 36\times10^7\,[\mathrm{mg}] = \left(48 - 24e^{-t/(24\times10^3)}\right)\times10^7$$

$$\begin{aligned}&\implies 36 = 48 - 24e^{-t/(24\times10^3)}\\ &\implies 12 = 24e^{-t/(24\times10^3)}\\ &\implies \frac{t}{24\times10^3} = -\ln\frac{1}{2} = \ln 2\\ &\implies \boxed{t = (24\times10^3)\ln 2 \approx 16.6\times10^3\,\text{min} \approx 11.6\,\text{days}}\end{aligned}$$