Thompson Rivers University

MATH 2240
Differential Equations I

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MIDTERM EXAM #1
SOLUTIONS

17 February 2012 08:30–09:20

PROBLEM	GRADE	OUT OF
1		10
2		10
3		10
4		10
5		10
TOTAL:		50

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

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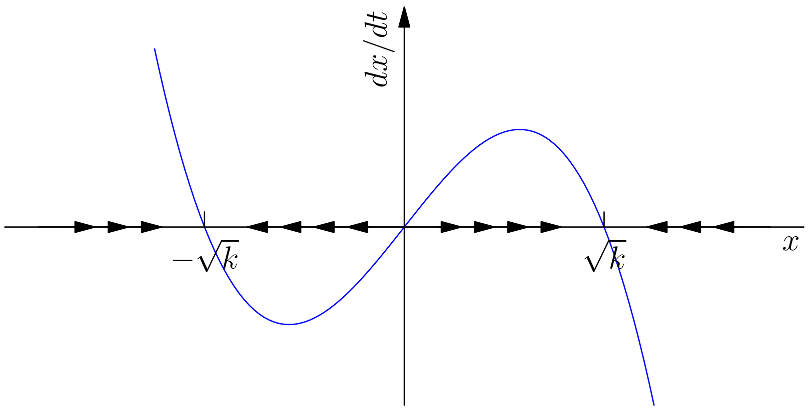
Problem 1: Consider the following differential equation in which $k > 0$ is a parameter:

$$\frac{dx}{dt} = kx - x^3$$

(a) Find the equilibrium solutions and classify them as to their stability. Sketch the phase portrait.

at equilibrium:

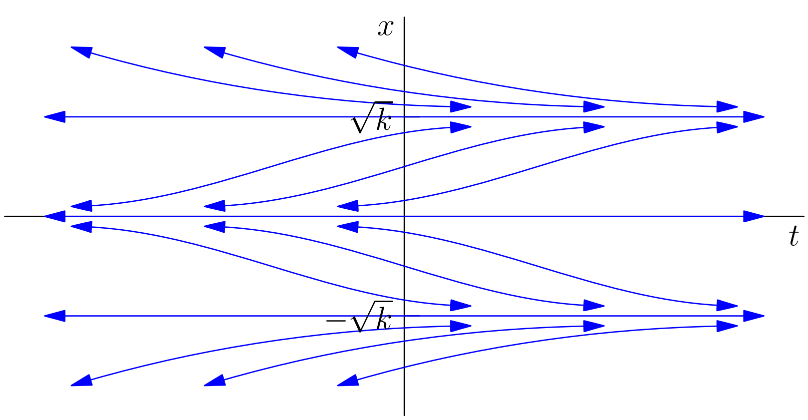
$$\frac{dx}{dt} = 0 = x(k - x^2) \implies x = 0 \pm \sqrt{k}$$



$x = \pm\sqrt{k}$ are asymptotically stable equilibria

$x = 0$ is un unstable equilibrium

(b) Sketch several typical solutions in the (x, t)-plane.



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Problem 2: Solve the initial value problem

$$\frac{dy}{dx} = 3x^2(y^2+1), \qquad y(0) = 1.$$

This is a separable equation:

$$\int \frac{dy}{y^2+1} = \int 3x^2\,dx$$

$$\implies \arctan y = x^3 + C$$

This is a convenient time to impose the initial conditions:

$$y(0) = 1 \implies \arctan 1 = 0^3 + C \implies C = \frac{\pi}{4}$$

Thus we have

$$\arctan y = x^3 + \frac{\pi}{4} \implies \boxed{y = \tan\left(x^3 + \frac{\pi}{4}\right)}$$

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Problem 3: Find an integrating factor and the general solution $y(x)$ for the following differential equation:

$$y' = 1 + x + y + xy.$$

This is a first-order linear DE, which we can write in "standard form" as

$$y' \underbrace{-(1+x)}_{P(x)} y = 1 + x.$$

An integrating factor is given by

$$\mu(x) = e^{\int P(x)\,dx} = e^{-\int (1+x)\,dx} = e^{-(x+\frac{x^2}{2})}.$$

Multiplying both sides of the DE by μ gives

$$\underbrace{y'e^{-(x+\frac{x^2}{2})} - (1+x)e^{-(x+\frac{x^2}{2})}y}_{\frac{d}{dx}\left(ye^{-(x+\frac{x^2}{2})}\right)} = (1+x)e^{-(x+\frac{x^2}{2})}$$

Integrating both sides then gives

$$ye^{-(x+\frac{x^2}{2})} = \int (1+x)e^{-(x+\frac{x^2}{2})}\,dx = -e^{-(x+\frac{x^2}{2})} + C$$

$$\implies \boxed{y = -1 + Ce^{x+\frac{x^2}{2}}; \quad C \in \mathbb{R}}$$

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Problem 4: Find the general solution $y(x)$:

$$y''' + y'' = e^{-x}.$$

method 1: This is a constant-coefficient linear DE, so first solve the corresponding homogeneous DE:

$$y''' + y'' = 0$$

by assuming $y_h = e^{rx}$:

$$\implies 0 = r^3 + r^2 = r^2(r+1) \implies r = -1 \text{ and } r = 0 \text{ (a repeated root of multiplicity 2)}$$

$$\begin{aligned} \implies y_h &= c_1 e^0 + c_2 x e^0 + c_3 e^{-x} \\ &= c_1 + c_2 x + c_3 e^{-x} \end{aligned}$$

Since the rhs e^{-x} satisfies the homogeneous equation, we must assume a particular solution of the form

$$\begin{aligned} y_p &= Axe^{-x} \\ \implies y_p' &= Ae^{-x} - Axe^{-x} \\ \implies y_p'' &= -2Ae^{-x} + Axe^{-x} \\ \implies y_p''' &= 3Ae^{-x} - Axe^{-x} \end{aligned}$$

Requiring that this satisfy the given DE yields:

$$(3Ae^{-x} - Axe^{-x}) + (-2Ae^{-x} + Axe^{-x}) = e^{-x}$$

$$\implies Ae^{-x} = e^{-x} \implies A = 1.$$

Thus $y_p = xe^{-x}$ furnishes a particular solution, and the general solution is

$$y = y_h + y_p = \boxed{c_1 + c_2 x + c_3 e^{-x} + xe^{-x}; \quad c_1, c_2, c_3 \in \mathbb{R}}$$

method 2:
The substitution $u = y''$ reduces the DE to first-order:

$$u' + u = e^{-x}$$

Using the integrating factor $\mu(x) = e^x$ gives

$$\underbrace{u'e^x + ue^x}_{\frac{d}{dx}(ue^x)} = 1 \implies ue^x = \int 1\,dx = x + C \implies u = y'' = xe^{-x} + Ce^{-x}$$

Now we can find y by integrating twice:

$$\begin{aligned} y' &= \int y''\,dx = \int (xe^{-x} + Ce^{-x})\,dx \\ &= -(x+1)e^{-x} - Ce^{-x} + D \\ \implies y &= \int [-(x+1)e^{-x} - Ce^{-x} + D]\,dx \\ &= \boxed{xe^{-x} + 2e^{-x} + Ce^{-x} + Dx + E} \end{aligned}$$

Rewriting this with $c_1 = E$, $c_2 = D$, $c_3 = C + 2$ gives the same answer as above.

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Problem 5: Kamloops Lake has a volume of $3.7\,\text{km}^3$. Water flows into the lake at a rate of $0.5\,\text{km}^3/\text{yr}$. Well-mixed water flows out of the lake at this same rate.

At time $t = 0$ the lake contains contaminant X at a concentration of $10\,\text{kg}/\text{km}^3$. Due to recently introduced environmental regulations, the water flowing into the lake contains contaminant X at a lower concentration of $5\,\text{kg}/\text{km}^3$.

(a) Find the amount (in kg) of contaminant X in the lake after t years.

Let $y(t)$ be the amount (in kg) of contaminant X in the lake.

Contaminant X flows into the lake at a constant rate of $(0.5\,\text{km}^3/\text{yr}) \times (5\,\text{kg}/\text{km}^3) = 2.5\,\text{kg}/\text{yr}$

Contaminant X flows out of the lake at an instantaneous rate of $(0.5) \times \left(\frac{y}{3.7}\right) = \frac{y}{7.4}$.

Thus the rate of change of y is given by

$$\frac{dy}{dt} = \text{“rate in”} - \text{“rate out”} = 2.5 - \frac{y}{7.4} \quad \Longrightarrow \quad \frac{dy}{dt} + \frac{y}{7.4} = 2.5$$

We can solve this DE using the integrating factor $\mu(t) = e^{t/7.4}$:

$$\underbrace{\frac{dy}{dt}e^{t/7.4} + \frac{y}{7.4}e^{t/7.4}}_{\frac{d}{dt}(ye^{t/7.4})} = 2.5e^{t/7.4} \Longrightarrow ye^{t/7.4} = \int 2.5e^{t/7.4}\,dt = (2.5)(7.4)e^{t/7.4} + C$$

$$\Longrightarrow y = 18.5 + Ce^{-t/7.4}$$

Imposing initial conditions:

$$y(0) = (3.7)(10) = 18.5 + C \Longrightarrow C = 18.5$$

$$\Longrightarrow \boxed{y(t) = 18.5(1 + e^{-t/7.4})}$$

(b) How long will it take for the concentration of contaminant X in the lake to fall to $6\,\text{kg}/\text{km}^3$?

$$y(T) = (3.7)(6) = 18.5(1 + e^{-T/7.4})$$

$$\Longrightarrow e^{-T/7.4} = \frac{22.2}{18.5} - 1 = 0.2$$

$$\Longrightarrow \boxed{T = -7.4 \ln 0.2 \approx 11.9\,\text{years}}$$