Name: ______________________ Student #: ______________

THOMPSON RIVERS UNIVERSITY

MATH 2240
Differential Equations 1

Instructor: Richard Taylor

FINAL EXAM

18 April 2015 14:00–17:00

PROBLEM	GRADE	OUT OF
1		10
2		12
3		7
4		9
5		14
6		9
7		7
8		10
TOTAL:		78

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 9 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

/10

Problem 1: In fisheries management the differential equation

$$\frac{dx}{dt} = kx(M - x) - h$$

is used to model a fish population x as a function of time t. The constants k, M are positive; the constant $h > 0$ is the *harvesting rate* at which fish are removed by fishing. Assume $h < kM^2/4$.

(a) Find the equilibrium solutions (expressed in terms of k, M and h).

/2

(b) Classify the equilibrium solutions in terms of stability.

/2

(c) Sketch typical solutions in the (t, x)-plane.

/2

(d) The model predicts that the fish population will go extinct if the initial population is less than x_{min}. Find x_{min} in terms of k, M and h.

/2

(e) What happens if the harvesting rate is high, with $h > hM^2/4$? Describe the qualitative solutions in this case. What are the implications for the fish population?

/2

/12 **Problem 2:** Solve the following first-order differential equations:

/4 (a) $\dfrac{dy}{dx} = 6e^{2x-y}, \quad y(0) = 0$

/4 (b) $x^3 + \dfrac{y}{x} + \left(y^2 + \ln x\right) \dfrac{dy}{dx} = 0$

/4 (c) $xy' = 2y + x^2 \cos x$

/7 **Problem 3:** Consider the initial value problem

$$\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 0.$$

(a) Explain why this is a difficult problem to solve analytically (i.e. exactly).

/2

(b) Use Euler's method with step size $h = 0.2$ to approximate $y(0.6)$.

/5

/9 **Problem 4:** Consider the second-order equation

$$x^2y'' - xy' + y = 0 \qquad (x > 0).$$

(a) Verify that $y_1(x) = x$ and $y_2(x) = x \ln x$ are solutions of this equation.

/2

(b) What is the general solution of this equation?

/2

(c) Using linear independence, carefully justify why your answer to (b) gives *every* possible solution of the given equation.

/5

/14 **Problem 5:** Solve the following:

/5 (a) $y'' + 9y = 2\cos 3x + 3\sin 3x$

/4 (b) $y^{(4)} + 2y'' = 0.$

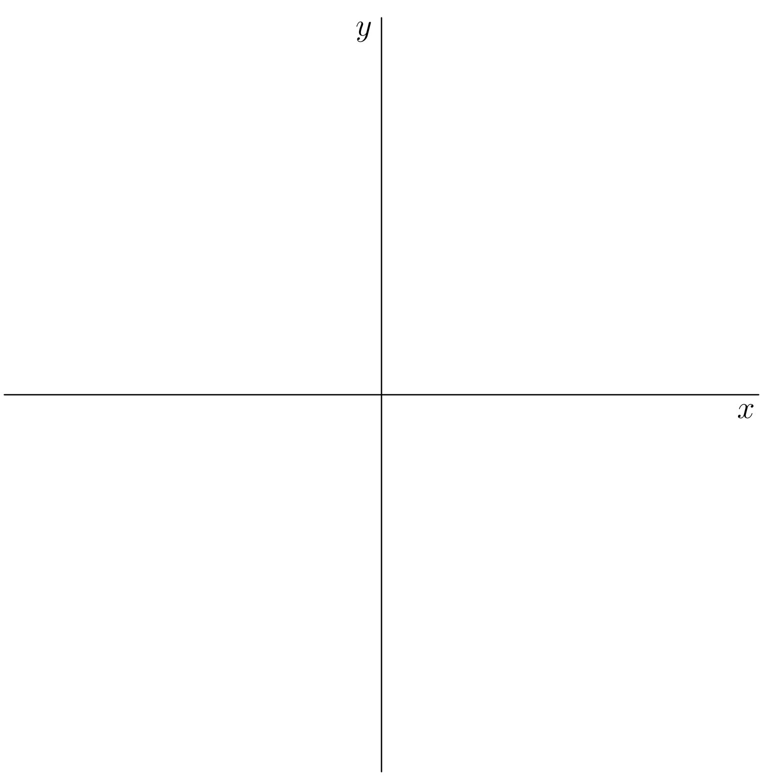
/5 (c) $y'' - 5y' + 6y = f(x)$ (express your answer in terms of $f(x)$)

/9 **Problem 6:** Consider the linear system $\begin{cases} x' = 2x + 3y \\ y' = 2x + y \end{cases}$

/5 (a) Find the general solution of the system.

/2 (b) Find the solution that satisfies the initial conditions $x(0) = 11$, $y(0) = -1$.

/2 (c) Sketch the phase portrait (qualitative solutions in the (x, y)-plane).



/7 **Problem 7:** Consider the matrix $A = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$.

/4 (a) Use a matrix power series to show that $e^{At} = \begin{bmatrix} 1+6t & 4t \\ -9t & 1-6t \end{bmatrix}$.

/3 (b) Use the result from part (a) to solve the following initial value problem:

$$\begin{cases} x' = 6x + 4y & \qquad x(0) = 1 \\ y' = -9x - 6y & \qquad y(0) = 2 \end{cases}$$

/10 **Problem 8:** Consider the nonlinear system $\begin{cases} x' = x - y \\ y' = 1 - x^2 \end{cases}$

(a) Find all the equilibria of the system and classify them according to type and stability.

/6

(b) Sketch the phase portrait (qualitative solutions in the (x, y)-plane).

/4