THOMPSON RIVERS UNIVERSITY

MATH 2240
Differential Equations I

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FINAL EXAM
SOLUTIONS

16 April 2012 14:00–17:00

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 8 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		12
3		8
4		8
5		8
6		11
7		12
TOTAL:		71

$\big/12$ **Problem 1:** Find the most general solution $y(x)$ for each of the following:

$\big/4$ (a) $xy' + 4y = x^3 - x$

This is a linear first-order DE. In standard form:

$$y' + \frac{4}{x}y = x^2 - 1$$

this gives the integrating factor $\mu(x) = e^{\int 4/x\,dx} = e^{4\ln x} = e^{\ln x^4} = x^4$. Multiplying by μ:

$$\underbrace{x^4y' + 4x^3y}_{\frac{d}{dx}(x^4y)} = x^6 - x^4 \implies x^4y = \int x^6 - x^4\,dx$$
$$= \tfrac{1}{7}x^7 - \tfrac{1}{5}x^5 + C$$

$$\implies \boxed{y = \tfrac{1}{7}x^3 - \tfrac{1}{5}x + \frac{C}{x^4} \quad (C \in \mathbb{R})}$$

$\big/4$ (b) $y''' - 5y'' + 3y' + 9y = 0$

A constant-coefficient linear 3rd-order DE. Assuming $y = e^{rx}$ gives the characteristic polynomial

$$P(r) = r^3 - 5r^2 + 3r + 9$$

By guess-and-check we find that $r = -1$ is a root (the Rational Roots Theorem furnishes good guesses $\pm1, \pm3$ as the only possible rational roots). So $(r+1)$ is factor of $P(r)$, and by polynomial long-division:

$$P(r) = (r+1)(r^2 - 6r + 9) = (r+1)(r-3)^3$$

so the roots are $r = -1$ and $r = 3$ (repeated, with multiplicity 2). Thus the general solution is

$$\boxed{y(x) = c_1e^{3x} + c_2xe^{3x} + c_3e^{-x} \quad (c_1, c_2, c_3 \in \mathbb{R})}$$

$\big/4$ (c) $y' = 4x^3y^2 - y^2$

This one is separable:

$$y' = (4x^3 - 1)y^2 \implies \int \frac{dy}{y^2} = \int (4x^3 - 1)\,dx$$
$$\implies -y^{-1} = x^4 - x + C$$
$$\implies \boxed{y = \frac{1}{C + x - x^4} \quad (C \in \mathbb{R})}$$

/12 **Problem 2:** When certain kinds of chemicals are combined, the rate at which the new compound is formed is modeled by the autonomous differential equation

$$\frac{dx}{dt} = k(A-x)(B-x)$$

where $x(t)$ denotes the number of grams of the new compound present at time t. The quantities $k > 0$ and $B > A > 0$ are constants.

(a) Find the equilibrium (i.e. constant) solutions and their stability. Sketch the phase portrait.

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$x = A$ is an asymptotically stable equilibrium.

$x = B$ is an unstable equilibrium.

(b) Use your phase portrait to predict the behavior of $x(t)$ as $t \to \infty$.

/2

If $x(0) < B$ then $x(t) \to A$, otherwise $x(t) \to \infty$.

(c) Sketch typical solution curves in the (t, x)-plane.

/2

(d) Explain why the graphs of $x(t)$ in part (c) cannot cross the equilibrium solutions.

/2

Suppose some solution curve crosses an equilibrium solution at a point (t, x). This curve must have $x' \neq 0$ at this point, but since this point is also on an equilibrium (constant) solution curve, it must have $x' = 0$ ... a contradiction, since x' is single-values at every point in the (t, x)-plane.

(e) Consider the special case with $k = 1$ and $A = B$. Predict the behavior of $x(t)$ as $t \to \infty$.

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$x' = (A-x)^2$ has a single (semi-stable) equilibrium at $x = A$.

If $x(0) < A$ then $x(t) \to A$, otherwise $x(t) \to \infty$.

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Problem 3: Consider the following differential equation:

$$x^2y'' + xy' + y = 0 \qquad (x > 0)$$

(a) Show that $y_1(x) = \cos(\ln x)$ and $y_2(x) = \sin(\ln x)$ are both solutions of this equation.

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We have:

$$\begin{aligned} y_1' &= -\frac{1}{x}\sin(\ln x) \\ y_1'' &= \frac{1}{x^2}\sin(\ln x) - \frac{1}{x^2}\cos(\ln x) \end{aligned}$$

and substitution into the given DE yields

$$\begin{aligned} \implies x^2y_1'' + xy_1' + y_1 &= x^2\left[\frac{1}{x^2}\sin(\ln x) - \frac{1}{x^2}\cos(\ln x)\right] + x\left[-\frac{1}{x}\sin(\ln x)\right] + \cos(\ln x) \\ &= \sin(\ln x) - \cos(\ln x) - \sin(\ln x) + \cos(\ln x) \\ &\equiv 0 \quad \surd \end{aligned}$$

so y_1 is indeed as solution of the given DE.

Similarly,

$$\begin{aligned} y_2' &= \frac{1}{x}\cos(\ln x) \\ y_2'' &= -\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x) \end{aligned}$$

and substitution into the DE yields

$$\begin{aligned} \implies x^2y_2'' + xy_2' + y_2 &= x^2\left[-\frac{1}{x^2}\cos(\ln x) - \frac{1}{x^2}\sin(\ln x)\right] + x\left[\frac{1}{x}\cos(\ln x)\right] + \sin(\ln x) \\ &= -\cos(\ln x) - \sin(\ln x) + \cos(\ln x) + \sin(\ln x) \\ &\equiv 0 \quad \surd \end{aligned}$$

so y_2 is indeed as solution of the DE.

(b) Write the most general solution of this equation, and *prove* that your solution is indeed the most general solution possible.

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Since the DE is linear 2nd-order, the most general solution is

$$y = c_1y_1 + c_2y_2$$

where y_1, y_2 are *any* two linearly independent solutions. By considering the Wronskian:

$$\begin{aligned} W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} \\ &= \frac{1}{x}cos^2(\ln x) + \frac{1}{x}\sin^2(\ln x) \\ &= \frac{1}{x}\left(cos^2(\ln x) + \sin^2(\ln x)\right) = \frac{1}{x} \neq 0 \quad \forall x > 0 \end{aligned}$$

we find that y_1, y_2 are indeed linearly independent, so the most general solution is an arbitrary linear combination:

$$\boxed{y(x) = c_1\cos(\ln x) + c_2\sin(\ln x) \quad (c_1, c_2 \in \mathbb{R})}$$

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Problem 4: Solve the following initial-value problem:

$$y'' + 4y = 3\sin 2x, \qquad y(0) = 3, \qquad y'(0) = -\tfrac{3}{4}$$

First consider the corresponding homogeneous equation $y_h'' + 4y_h = 0$ where assuming $y = e^{rx}$ gives the characteristic polynomial

$$r^4 + 4 = 0 \implies r = \pm 2i \implies y_h = c_1\cos(2x) + c_2\sin(2x)$$

Now use undetermined coefficients to find a particular solution. Since $f(x) = 3\sin 2x$ is a solution of the homogeneous problem, a suitable form for the particular solution is

$$y_p = x(A\cos 2x + B\sin 2x).$$

This gives:

$$\begin{aligned}
&\implies y_p' = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos 2x \\
&\implies y_p'' = -2A\sin 2x - 2A\sin 2x - 4Ax\cos 2x + 2B\cos 2x + 2B\cos 2x - 4Bx\sin 2x \\
&\qquad\quad = -4A\sin 2x + 4B\cos 2x - 4Ax\cos 2x - 4Bx\sin 2x
\end{aligned}$$

Substitution into the DE yields

$$(-4A\sin 2x + 4B\cos 2x - 4Ax\cos 2x - 4Bx\sin 2x) + 4(Ax\cos 2x + Bx\sin 2x) = 3\sin 2x$$

$$\implies 4B\cos 2x - 4A\sin 2x = 3\sin 2x$$

$$\implies \begin{cases} 4B = 0 \\ -4A = 3 \end{cases} \implies A = -\frac{3}{4}, \quad B = 0$$

So the general solution is

$$y = c_1\cos 2x + c_2\sin 2x - \frac{3}{4}x\cos 2x.$$

Now to impose initial conditions...

$$\begin{aligned}
3 &= y(0) = c_1\cos 0 + c_2\sin 0 - 0 = c_1 \implies c_1 = 3 \\
-\frac{3}{4} &= y'(0) = -2c_1\sin 0 + 2c_2\cos 0 - \frac{3}{4}\cos 0 + \frac{3}{2}\cdot 0\sin 0 = 2c_2 - \frac{3}{4} \implies c_2 = 0
\end{aligned}$$

and so

$$\boxed{y = 3\cos 2x - \frac{3}{4}x\cos 2x}$$

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Problem 5: Find the general solution $y(x)$ of the following:

$$y'' + 2y' + y = e^{-x} \ln x$$

First consider the corresponding homogeneous equation $y_h'' + 2y_h' + y_h = 0$, where assuming $y = e^{rx}$ gives the characteristic polynomial

$$0 = r^2 + 2r + 1 = (r+1)^2 \implies r = -1 \text{ (repeated root).}$$

So two linearly independent solutions are

$$y_1 = e^{-x} \quad \text{and} \quad y_2 = xe^{-x}.$$

Now use variation of parameters to find a particular solution, i.e. assume a solution of the form

$$y = u_1 y_1 + u_2 y_2$$

where the functions $u_1(x)$, $u_2(x)$ must satisfy

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = e^{-x} \ln x \end{cases} \implies \begin{cases} u_1' e^{-x} + u_2' x e^{-x} = 0 \\ -u_1' e^{-x} + u_2' (e^{-x} - xe^{-x}) = e^{-x} \ln x \end{cases}$$

Adding these equations yields

$$u_2' e^{-x} = e^{-x} \ln x \implies u_2' = \ln x \implies u_2 = \int \ln x \, dx = x \ln x - x.$$

Then

$$u_1' = -x u_2' = -x \ln x \implies u_1 = -\int x \ln x \, dx = x^2 \left(\tfrac{1}{4} - \tfrac{1}{2} \ln x \right).$$

So the general solution is

$$y = c_1 e^{-x} + c_2 x e^{-x} + x^2 \left(\tfrac{1}{4} - \tfrac{1}{2} \ln x \right) e^{-x} + (x \ln x - x)\, x e^{-x}$$

$$\implies \boxed{y = c_1 e^{-x} + c_2 x e^{-x} + x^2 \left(\tfrac{1}{2} \ln x - \tfrac{3}{4} \right) e^{-x}}$$

/11 **Problem 6:** Consider the following system of differential equations: $\begin{cases} x' = 2x + 4y \\ y' = -x + 6y \end{cases}$

(a) Find the most general solution $(x(t), y(t))$ of this system.

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Write the system as $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$. Eigenvalues of A are given by

$$0 = \det(A - rI) = \begin{vmatrix} 2-r & 4 \\ -1 & 6-r \end{vmatrix} = (2-r)(6-r) + 4 = r^2 - 8r + 16 = (r-4)^2 \implies r = 4.$$

The corresponding eigenvector $\mathbf{v}$ is given by

$$(A - 4I)\mathbf{v} = \mathbf{0} \implies \begin{bmatrix} -2 & 4 & 0 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

To form a second, linearly independent solution we need a generalized eigenvector $\mathbf{b}$ given by

$$(A - 4I)\mathbf{b} = \mathbf{v} \implies \begin{bmatrix} -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \text{ (for instance).}$$

Now we can form the general solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right) e^{4t}$$

or in terms of the component functions:

$$\boxed{\begin{array}{ll} x(t) = (2c_1 - c_2 + 2c_2 t)e^{4t} & \\ & (c_1, c_2 \in \mathbb{R}) \\ y(t) = (c_1 + c_2 t)e^{4t} & \end{array}}$$

(b) Find the solution that satisfies the initial conditions $x(0) = 3$, $y(0) = 2$.

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$$\begin{cases} 3 = x(0) = 2c_1 - c_2 \\ 2 = y(0) = c_1 \end{cases} \implies c_1 = 2,\ c_2 = 1$$

$$\boxed{\begin{array}{l} x(t) = (3 + 2t)e^{4t} \\ y(t) = (2 + t)e^{4t} \end{array}}$$

/12 **Problem 7:** Raleigh's differential equation (with constant $k \in \mathbb{R}$) is given by

$$x'' + k\left(\tfrac{1}{3}(x')^3 - x'\right) + x = 0$$

/2 (a) Show how to transform this equation into the equivalent system $\begin{cases} x' = y \\ y' = -k\left(\tfrac{1}{3}y^3 - y\right) - x. \end{cases}$

$$\text{Let } y = x' \implies \begin{cases} x' = y \\ y' = x'' = -k\left(\tfrac{1}{3}(x')^3 - x'\right) - x = -k\left(\tfrac{1}{3}y^3 - y\right) - x \end{cases}$$

/2 (b) Show that $(0,0)$ is the only equilibrium solution of your system in part (a).

At equilibrium:

$$\begin{aligned} 0 = x' = y &\implies y = 0 \\ 0 = y' = -k\left(\tfrac{1}{3}0^3 - 0\right) - x &\implies x = 0 \end{aligned}$$

$$\implies (x, y) = (0, 0)$$

/3 (c) Show that $(0,0)$ is unstable when $k > 0$. For what value(s) of k is $(0,0)$ an unstable spiral point?

Near $(0,0)$ the cubic term in y is negligible, and linearization near this point gives

$$\begin{cases} x' = y \\ y' = -x + ky \end{cases} \quad \text{or in matrix form: } \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & k \end{bmatrix} \mathbf{x}$$

The coefficient matrix has $\Delta = 1$ and $\tau = k$. Since $\Delta > 0$, stability is determined by the sign of τ. In particular, for $\tau = k > 0$ the origin is unstable.

$(0,0)$ is a spiral point if

$$0 < 4\Delta - \tau^2 = 4 - k^2 \implies k^2 < 4 \implies -2 < k < 2.$$

Together with the condition for instability ($k > 0$) this gives $\boxed{0 < k < 2}$ for the origin to be an unstable spiral.

/3 (d) Show that $(0,0)$ is stable when $k < 0$. For what value(s) of k is $(0,0)$ a stable spiral point?

As above, stability is determined by the sign of τ. In particular, for $\tau = k < 0$ the origin is asymptotically stable.

Together with the condition for $(0,0)$ to be a spiral point ($-2 < k < 2$) this gives $\boxed{-2 < k < 0}$ for the origin to be a stable spiral.

/2 (e) Classify the stability type of $(0,0)$ when $k = 0$, and sketch the phase portrait for this case.

For $k = 0$ the system is linear, with coefficient matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. In this case we have $\Delta = 1$ and $\tau = 0$ so $(0,0)$ is a *center*. Phase trajectories are closed (clockwise) circles about the origin.