THOMPSON RIVERS UNIVERSITY

MATH 2650
Calculus 3 for Engineering

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MIDTERM EXAM #1
SOLUTIONS

17 Oct 2018 13:00–14:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved formula sheet.
8. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		3
2		3
3		4
4		8
5		6
6		6
TOTAL:		30

/3 **Problem 1:** Write a chain rule for $\frac{\partial w}{\partial s}$ where

$$w = g(x,y), \qquad x = h(r,s,t), \qquad y = k(r,s,t).$$

$$\frac{\partial w}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial k}{\partial s}$$

/3 **Problem 2:** Determine a vector that is normal to the surface $x^3 + y^2 + z^3 = 0$ at the point $(1,0,-1)$.

This is a level surface of $g(x,y,z) = x^3 + y^2 + z^3$ so

$$\nabla g = (3x^2, 2y, 3z^2)$$

is normal to the surface (the gradient is always $\perp$ to a level curve/surface). At the point $(1,0,-1)$ we have

$$\nabla g(1,0,-1) = \boxed{(3,0,3)}$$

/4 **Problem 3:** Find a linear approximation of $f(x,y) = e^{2y-x}$ for (x,y) near the point $(1,2)$.

$$\begin{array}{l} f_x = -e^{2y-x} \\ f_y = 2e^{2y-x} \end{array} \implies \begin{array}{l} f_x(1,2) = -e^3 \\ f_y(1,2) = 2e^3 \\ f(1,2) = e^3 \end{array}$$

$$\begin{aligned} L(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= e^3 + (-e^3)(x-1) + (2e^3)(y-2) = \boxed{e^3(-2-x+2y)} \end{aligned}$$

/8 **Problem 4:** Consider the function $f(x,y,z) = \ln(x^2+y^2-1)+y+6z$ and the point $P(1,1,0)$.

(a) Evaluate ∇f at P.

/2

$$\nabla f = \left(\frac{2x}{x^2+y^2-1}, \frac{2y}{x^2+y^2-1}+1, 6\right) \implies \nabla f(1,1,0) = \boxed{(2,3,6)}$$

(b) Evaluate the derivative of f at P in the direction of the vector $(0,1,1)$.

/2

Form a unit vector $\mathbf{u}$ in the direction of $\mathbf{v} = (0,1,1)$:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(0,1,1)}{\sqrt{2}}.$$

Then

$$\begin{aligned} D_{\mathbf{u}}f(1,1,0) &= \nabla f(1,1,0)\cdot\mathbf{u} \\ &= (2,3,6)\cdot\frac{(0,1,1)}{\sqrt{2}} \\ &= \boxed{\frac{9}{\sqrt{2}} = \frac{9}{2}\sqrt{2}} \end{aligned}$$

(c) In what direction does $f(x,y,z)$ increase most rapidly at P?

/2

In the direction of $\boxed{\nabla f(1,1,0) = (2,3,6)}$

(d) Evaluate the derivative of f in the direction of fastest increase of f at P.

/2

In the direction of $\nabla f(1,1,0)$ the directional derivative is

$$|\nabla f(1,1,0)| = |(2,3,6)| = \sqrt{2^2+3^2+6^2} = \sqrt{49} = \boxed{7}$$

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Problem 5: Find all the local minima, maxima and saddle points of the function

$$f(x,y) = x^2 + xy + 3x + 2y + 5.$$

$$\begin{cases} 0 = f_x = 2x + y + 3 \\ 0 = f_y = x + 2 \end{cases}$$

$$x = -2 \implies 2(-2) + y + 3 = 0 \implies y = 1$$

so there is just one critical point, at $(-2, 1)$.

To apply the 2nd derivative test we evaluate

$$\begin{matrix} f_{xx} = 2 \\ f_{yy} = 0 \\ f_{xy} = 1 \end{matrix} \implies D = f_{xx}f_{yy} - [f_{xy}]^2 = -1 < 0.$$

Thus $(-2, 1)$ is a saddle point. There are no local minima or maxima.

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Problem 6: Use the method of Lagrange multipliers to find the points on the curve $x^2 + xy + y^2 = 1$ that are nearest and farthest from the origin. (Hint: you can do this by locating the minimum and maximum of $f(x, y) = x^2 + y^2$.)

We need to find the minimum and maximum of

$$f(x, y) = x^2 + y^2$$

subject to the constraint

$$g(x, y) = x^2 + xy + y^2 = 1.$$

We have

$$\begin{aligned} \nabla f &= (2x, 2y) \\ \nabla g &= (2x + y, x + 2y). \end{aligned}$$

Applying the method of Lagrange multipliers, we need to solve the following system for unknowns (x, y, λ):

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \implies \begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(x + 2y) \\ x^2 + xy + y^2 = 1. \end{cases}$$

This gives

$$\begin{aligned} \lambda = \frac{2x}{2x + y} = \frac{2y}{x + 2y} &\implies 2x(x + 2y) = 2y(2x + y) \\ &\implies 2x^2 + 4xy = 4xy + 2y^2 \\ &\implies x^2 = y^2 \\ &\implies y = \pm x. \end{aligned}$$

Case $y = x$:

$$x^2 + xy + y^2 = 1 \implies x^2 + x(x) + (x)^2 = 3x^2 = 1 \implies x = \pm\frac{1}{\sqrt{3}}$$

This gives the points $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. At both of these points the squared distance to the origin is

$$f\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right) = \sqrt{\frac{1}{3} + \frac{1}{3}} = \frac{\sqrt{2}}{\sqrt{3}}.$$

Case $y = -x$:

$$x^2 + xy + y^2 = 1 \implies x^2 + x(-x) + (-x)^2 = x^2 = 1 \implies x = \pm 1.$$

This gives the points $(1, -1)$ and $(-1, 1)$. At both of these points the squared distance to the origin is

$$f(\pm 1, \pm 1) = \sqrt{1 + 1} = \sqrt{2}.$$

Thus the nearest points to the origin are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. The farthest points are $(1, -1)$ and $(-1, 1)$.