MATH 211 – Review Problems

23 Nov. 2007

1. Find the angle between the diagonal of a cube and one of its edges.

2. For given vectors $\mathbf{x}$ and $\mathbf{y}$, let $a = |\mathbf{x}|$ and $b = |\mathbf{y}|$. Show that the vectors $a\mathbf{y} + b\mathbf{x}$ and $b\mathbf{x} - a\mathbf{y}$ are orthogonal.

3. Let $\mathbf{x} = (2, 1, -2)$. Find a unit vector $\mathbf{u}$ so that $\mathbf{x} \cdot \mathbf{u}$ is a maximum.

4. Show that the points $(3, 5, 6)$, $(1, 2, 7)$ and $(6, 1, 0)$ are the vertices of a right triangle.

5. Find the angle between the diagonal of a cube and the diagonal of one of its faces.

6. At time t the location of particle A is given by $\mathbf{r}_1(t) = (-6, 8) + t(10, -3)$ and the location of particle B is given by $\mathbf{r}_2 = (0, 1) + t(1, 1)$. Show clearly that the paths of the particles intersect one another, but the particles do not collide.

7. Show that the lines $\mathbf{r}_1(t) = (1, 2, 3) + t(2, 1, 2)$ and $\mathbf{r}_2(t) = (3, 2, 0) + s(-4, 1, 1)$ do not intersect, yet are not parallel.

8. Graph the curve $\mathbf{r}(t) = (t^2, t^3/3)$ for $-3 \leq t \leq 3$.

9. Write a parametric vector equation for the curve of intersection of the plane $y - 2z = 0$ and the graph of $y = x^2$.

10. Write a parametric vector equation for the curve of intersection of the plane $y - z = 0$ and the sphere $x^2 + y^2 + z^2 = 9$.

11. A particle travels on the path given by $\mathbf{r}(t) = (t^2, t^3 - 4t)$ with $-3 \leq t \leq 3$. (a) Find at what time(s) and at what point(s) the particle crosses its own path. (b) Find the angle of intersection of the particle(s) path with itself.

12. The normal plane to a curve at a point is the plane which is perpendicular to the tangent line to the curve at that point. Find an equation for the normal plane to the curve $\mathbf{r}(t) = (t^2, \ln t, t^3 + 1)$ at the point $(1, 0, 2)$.

13. A particle leaves the path $\mathbf{r}(t) = (t, t^2, -t^3)$ at $t = 2$ and continues moving with the same velocity it had at that time. At what time and at what point will it intersect the plane $3x + 2y + z = 0$?

14. Find the equations of the tangent line and the normal plane to the curve $\mathbf{r}(t) = (t^2 + 1, t^2 - 1, t)$ at the point $(2, 0, 1)$.

15. Find the unit tangent vectors to the curve $\mathbf{r}(t) = (t, t^2, t^3)$ at the points where it intersects the plane $z = 2x + y$.

16. Prove that if $\mathbf{x}(t)$ is a unit vector then $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ are orthogonal.

17. Find the length of the helix defined by $\mathbf{r}(t) = (2\cos t, 2\sin t, 3t)$ for $0 \leq t \leq 4\pi$. Do this question in two ways, one of which does not use calculus.

18. Find the length of the path defined by $\mathbf{r}(t) = (t, \sqrt{3}t^2, 2t^3)$ for $0 \leq t \leq 3$.

19. For the space curve $\mathbf{r}(t) = (\frac{1}{2}t^2, t, 0)$ find the principal unit tangent vector and normal vector at $t = 2$. Sketch the curve and show $\mathbf{T}(2)$ and $\mathbf{N}(2)$.

20. For the space curve $\mathbf{r}(t) = (e^t \sin t, e^t \cos t, e^t)$ with $0 \leq t \leq 1$, find (a) a unit tangent vector at the point where $t = 0$, and (b) the length of the curve.

21. If $\mathbf{r}(t) = (e^t, e^{-t}, t\sqrt{2})$ show that the curvature function is $\kappa(t) = \sqrt{2}/(e^t + e^{-t})^2$.

22. Suppose that the temperature (in °C) at (x, y) (in cm) is given by $T(x, y) = (4x - 3y)$. (a) Sketch the level curves of T. (b) Find the hottest and coldest points on the circle $x^2 + y^2 = 25$. (c) If an ant is located at $(2, 1)$, give the direction the ant should move to: (i) increase its temperature most rapidly. (ii) decrease its temperature most rapidly. (ii) maintain constant temperature. (d) What rate of temperature change does the ant experience if it moves with velocity $(3, 7)$ cm/s? (e) What rate of temperature change does the ant experience if it moves toward the point $(5, 5)$?

23. The altitude z of a mountain is given by $z = 16 - x^2 - y^2$. (a) Make a contour map of the mountain by sketchig level curves of z. (b) If Antonino is at the point $(2, 1, 8)$ on the mountain, in what direction should he move to stay at the same altitude?

24. Suppose the pressure at the point (x, y) is given by $p(x, y) = 8 - x - 2y$. (a) Sketch the level curves of p. (b) A balloon is at the point $(-2, 1)$. In what direction should it be moved so that it (i) expands most rapidly? (ii) remains the same volume?

25. Let $f(x, y) = x^2 + 3y^2$. (a) Sketch the level curve $f(x, y) = 12$ and the gradient vector $\nabla f(3, 1)$. (b) Find an equation of the line tangent to the curve $f(x, y) = 12$ at the point $(3, 1)$.

26. If $g(x, y) = \ln(x^2 y)$, find: (a) the directional derivative of g at $(2, \frac{1}{4})$ in the direction $\mathbf{w} = (1, 2)$. (b) the direction in which the directional derivative at $(2, \frac{1}{4})$ is a maximum; give this maximum value. (c) an equation of the tangent plane to the graph of $z = g(x, y)$ at the point $(2, \frac{1}{4}, 0)$.

27. Find the directional derivative of $w(x, y, z) = 3x + xy + z^2 y$ at the point $(2, 3, -1)$ in the direction $(1, 1, 2)$.

28. The plane $y = x/2$ intersects the surface $z = x^2 + xy$ in a curve. Find a parametric vector equation of the line which is tangent to this curve at the point $(2, 1, 6)$.

29. Find an equation of the plane which is tangent to the surface $x^2 + y^2 + z^2 = 49$ at the point $(3, 2, 6)$.

30. Find an equation of the plane tangent to $z = 9 - x^2 - y^2$ at the point $(1, 2, 4)$.

31. Find equations for both the tangent plane and normal line to the surface $z = x^2 + y^3$ at the point $(3, 1, 10)$.

32. Draw the level curves of $z = y - x^2/4$ through the points $(0, 0)$, $(2, 3)$ and $(-2, -2)$. Draw ∇z at each of these points.

33. The area of a rectangle is given by $A = lw$. If w changes from 5 to 5.1 and l changes from 10 to 10.2 find: (a) the exact change in A. (b) an approximate change in A using a linearization of $A(l, w)$ at $(5, 10)$. (c) show geometrically what the difference between the two answers is.

34. Let $g(x, y) = 2x^2 y - 3x$, $x = 3t - 4$ and $y = t^2 + 1$. Find $\frac{dg}{dt}$ at $t = 2$ both directly and by using the chain rule.

35. The location of a particle at time t (in seconds) is given by $\mathbf{r}(t) = (t^2 - 1, t)$, where the unit of distance is cm. If the temperature (in °C) at (x, y) is given by $f(x, y) = xy^2$, find at $t = 2$: (a) the particle's rate of change of temperature per second. (b) the particle's rate of change of temperature per cm.

36. Find an equation of the plane tangent to the hyperboloid $x^2 + y^2 - z^2 = 18$ at the point $(3, 5, 4)$.

37. The temperature (in °C) at each point in space is given by $T(x, y, z) = (x^2 + z^2 y)^3$. A bee at the point $(2, 1, 1)$ flies in the direction in which it gets warmer as quickly as possible. In what direction does it fly?

38. The derivative of $f(x,y,z)$ at a given point P is greatest in the direction $(1,1,-1)$, and in this direction the value of its derivative is $2\sqrt{3}$. (a) find ∇f at P. (b) find the derivative of f at P in the direction $(1,1,0)$.

39. Find parametric vector equation for the line of intersection of the planes $x-y+z=1$ and $x+y-z=2$.

40. If the temperature (in °C) at (x,y,z) is $T = x^2+3y+2z$ (with x, y, z in cm) and the location of a particle at time t is $\mathbf{r}(t) = (t, t^2, t^3)$ find (at $t=1$) the rate of change of the particle's temperature: (a) per second. (b) per cm.

41. Find and classify the critical points of $g(x,y) = 3x^3+y^2-9x+4y$.

42. Find and classify the critical points of $f(x,y) = 4xy - x^4 - y^4 + 10$.

43. Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint $x^2+y^2=8$.

44. Find the maximum and minimum values of $f(x,y) = x-3y$ on the ellipse $4x^2+y^2=16$.

45. Evaluate $\iint_D (y+2)\,dA$ where $D = \{(x,y) : 0 \le x \le 9, -\sqrt{x} \le y \le \sqrt{x}\}$.

46. Evaluate $\iint_D (x^2+2y)\,dA$ where D is the region in the first quadrant bounded by the y-axis, $y=x^2$ and $y=9$.

47. Change the order of integration to evaluate $\displaystyle\int_0^1 \int_{y^2}^{y} \frac{y}{x}\,dx\,dy$.

48. Evaluate $\displaystyle\int_0^{25} \int_{\sqrt{y}}^{5} ye^{x^5}\,dx\,dy$.

49. Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{4-x^2}} e^{-x^2}e^{-y^2}\,dy\,dx$.

50. Evaluate $\iint_D (3x+2y)\,dA$ where D is the region in the xy-plane in the first quadrant between the circles $x^2+y^2=4$ and $x^2+y^2=25$.