THOMPSON RIVERS UNIVERSITY

MATH 211
Calculus III

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MIDTERM EXAM #2
SOLUTIONS

20 November 2009 09:30–10:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 4 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		4
2		6
3		5
4		4
5		8
TOTAL:		27

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Problem 1: The surface given by the graph of

$$x^2y + y^2z + z^2x = 2$$

passes through the point $(1, -1, 1)$. Find an equation of the tangent plane to the surface at this point.

The graph is a level surface of $f(x, y, z) = x^2y + y^2z + z^2x$. We have

$$\nabla f = (2xy + z^2, x^2 + 2yz, y^2 + 2zx)$$

which is normal to the level surface (and the tangent plane) at any point, so

$$\mathbf{n} = \nabla f(1, -1, 1) = (-1, -1, 3).$$

The equation of the tangent plane is therefore

$$\mathbf{x} \cdot \mathbf{n} = \mathbf{x}_0 \cdot \mathbf{n} \implies (x, y, z) \cdot (-1, -1, 3) = (1, -1, 1) \cdot (-1, -1, 3)$$

$$\implies \boxed{-x - y + z = 3}$$

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Problem 2: Find and classify the critical points of the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

The critical points are the solutions of

$$\nabla f = (3x^2 - 3y, 3y^2 - 3x) = (0, 0)$$

which gives

$$\begin{cases} 3x^2 - 3y = 0 \implies y = x^2 \\ 3y^2 - 3x = 0 \implies (x^2)^2 - x = 0 \implies x(x^3 - 1) = 0 \implies x = 0, 1 \end{cases}$$

Therefore the critical points are

$$\boxed{(0, 0) \text{ and } (1, 1).}$$

Classification:

$$\begin{cases} f_{xx} = 6x \\ f_{yy} = 6y \\ f_{xy} = -3 \end{cases} \implies D = f_{xx}f_{yy} - f_{xy}^2 = (6x)(6y) - (-3)^2 = 36xy - 9$$

At $(0, 0)$ we have $D = 0 - 9 = -9 < 0$ so $\boxed{(0, 0) \text{ is a saddle point.}}$

At $(1, 1)$ we have $D = 36 - 9 = 27 > 0$ and $f_{xx} = 6 > 0$ so $\boxed{(1, 1) \text{ is a local minimum.}}$

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Problem 3: Let $n \geq 1$ be a given integer. Find the maximum value of the function

$$f(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$$

subject to the constraint

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Let $g(x_1, \dots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$, then the constraint is $g = 1$. Lagrange multipliers gives

$$\nabla f = \lambda \nabla g \implies (1, 1, \dots, 1) = \lambda(2x_1, 2x_2, \dots, 2x_n) \implies x_1 = x_2 = \cdots = x_n = \frac{1}{2\lambda}.$$

Then the constraint $g = 1$ gives

$$1 = \underbrace{x_1^2 + x_1^2 + \cdots + x_1^2}_{n \text{ times}} = nx_1^2 \implies x_1 = x_2 = \cdots = x_n = \pm\frac{1}{\sqrt{n}}$$

The positive root gives the maximum of f:

$$\begin{aligned} f(x_1, \dots, x_n) &= x_1 + \cdots + x_n \\ &= \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = n\left(\frac{1}{\sqrt{n}}\right) = \boxed{\sqrt{n}} \end{aligned}$$

(The negative root gives the mimimum value $f = -\sqrt{n}$.)

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Problem 4: Evaluate the double integral

$$\iint_R (x^2 + y^2)\, dA$$

where R is the rectangle

$$R = \{(x, y) : 0 \leq x \leq a,\ 0 \leq y \leq b\}.$$

$$\begin{aligned} \iint_R (x^2 + y^2)\, dA &= \int_0^a \underbrace{\int_0^b (x^2 + y^2)\, dy}_{x^2 y + \frac{1}{3}y^3\Big|_{y=0}^{b} = x^2 b + \frac{1}{3}b^3} \quad dx \\ &= \int_0^a \left(x^2 b + \tfrac{1}{3}b^3\right) dx \\ &= \tfrac{1}{3}x^3 b + \tfrac{1}{3}b^3 x\Big|_0^a \\ &= \boxed{\tfrac{1}{3}a^3 b + \tfrac{1}{3}b^3 a} \end{aligned}$$

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Problem 5: Let a, b, c be given constants and consider the curve in $\mathbb{R}^3$ given by the graph of

$$\mathbf{r}(t) = (at^2, bt, c\ln t), \quad 1 \le t \le T.$$

(a) Find the unit tangent vector $\mathbf{T}$ to this curve at the point where $t = 1$.

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$$\mathbf{r}' = (2at, b, c/t) \implies \mathbf{r}'(1) = (2a, b, c);$$

$$|\mathbf{r}'| = \sqrt{(2a)^2 + (b)^2 + (c)^2} = \sqrt{4a^2 + b^2 + c^2}$$

$$\implies \mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \boxed{\frac{(2a, b, c)}{\sqrt{4a^2 + b^2 + c^2}}}$$

(b) Express the length of the curve as a definite integral (do not evaluate this integral).

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$$\begin{aligned} L &= \int_1^T |\mathbf{r}'(t)|\, dt \\ &= \int_1^T \sqrt{(2at)^2 + b^2 + \left(\frac{c}{t}\right)^2}\, dt \\ &= \boxed{\int_1^T \sqrt{4a^2t^2 + b^2 + \frac{c^2}{t^2}}\, dt} \end{aligned}$$

(c) Evaluate your integral from part (b) for the case where $b^2 = 4ac$.

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$$\begin{aligned} L &= \int_1^T \sqrt{4a^2t^2 + 4ac + \frac{c^2}{t^2}}\, dt \\ &= \int_1^T \sqrt{\frac{4a^2t^4 + 4act^2 + c^2}{t^2}}\, dt \\ &= \int_1^T \sqrt{\frac{(2at^2 + c)^2}{t^2}}\, dt \\ &= \int_1^T \frac{2at^2 + c}{t}\, dt \\ &= \int_1^T \left(2at + \frac{c}{t}\right)\, dt \\ &= at^2 + c\ln|t|\Big|_1^T = \boxed{aT^2 + c\ln T - a} \end{aligned}$$