THOMPSON RIVERS UNIVERSITY

MATH 211
Calculus III

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

16 November 2007 11:30–12:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 4 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		5
3		4
4		8
5		6
6		6
TOTAL:		34

/5

Problem 1: Find an equation for the plane through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

The normal $\mathbf{n}$ must be $\perp$ to both $(a, -b, 0)$ and $(a, 0, -c)$

$$\therefore \quad \mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & -b & 0 \\ a & 0 & -c \end{vmatrix} = (bc, ac, ab)$$

so, with $\mathbf{r}_0 = (a, 0, 0)$, the equation of the plane is

$$\begin{aligned} (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 &\implies \big((x, y, z) - (a, 0, 0)\big) \cdot (bc, ac, ab) = 0 \\ &\implies (x - a)bc + yac + zab = 0 \\ &\implies \boxed{bcx + acy + abz = abc} \\ &\implies \boxed{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1} \end{aligned}$$

/5

Problem 2: Find an equation for the line that is parallel to the intersection of the planes $3x + y + z = 5$ and $x - 2y + 3z = 1$, and passes through the point $(5, 2, 3)$.

$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ with $\mathbf{r}_0 = (5, 2, 3)$.

The direction $\mathbf{v}$ must be $\perp$ to both $(3, 1, 1)$ and $(1, -2, 3)$

$$\therefore \quad \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = (5, -8, -7)$$

so the equation of the line is

$$\boxed{\mathbf{r}(t) = (5, 2, 3) + t(5, -8, -7) = (5 + 5t, 2 - 8t, 3 - 7t)}$$

/4

Problem 3: Find the curvature function $\kappa(t)$ for the space curve $\mathbf{r}(t) = (t, 0, t^3)$.

$$\begin{aligned} \mathbf{r}'(t) &= (1, 0, 3t^2) \\ \mathbf{r}''(t) &= (0, 0, 6t) \end{aligned} \implies \mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 3t^2 \\ 0 & 0 & 6t \end{vmatrix} = (0, -6t, 0)$$

$$\implies \kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \boxed{\frac{|6t|}{(1 + 9t^4)^{3/2}}}$$

/8

Problem 4: The temperature T (in °C) at points in the xy-plane is given by the function

$$T(x,y) = x^2 + y^2.$$

(a) An ant walks along the curve with equation $x^2y = 16$. What is the lowest temperature that the ant could encounter?

/6

$$\text{Let } g(x,y) = x^2y = 16.$$

$$\text{Lagrange multipliers: } \nabla T = \lambda \nabla g \implies \begin{cases} 2x = \lambda 2xy & (1) \\ 2y = \lambda x^2 & (2) \\ x^2y = 16 & (3) \end{cases}$$

$(3) \implies x, y \neq 0$ so that $(1) \implies 2x(1-\lambda y) = 0 \implies \lambda = \dfrac{1}{y}$

then $(2) \implies 2y = \left(\dfrac{1}{y}\right)x^2 \implies x^2 = 2y^2$

then $(3) \implies (2y^2)y = 16 \implies y^3 = 8 \implies y = 2$

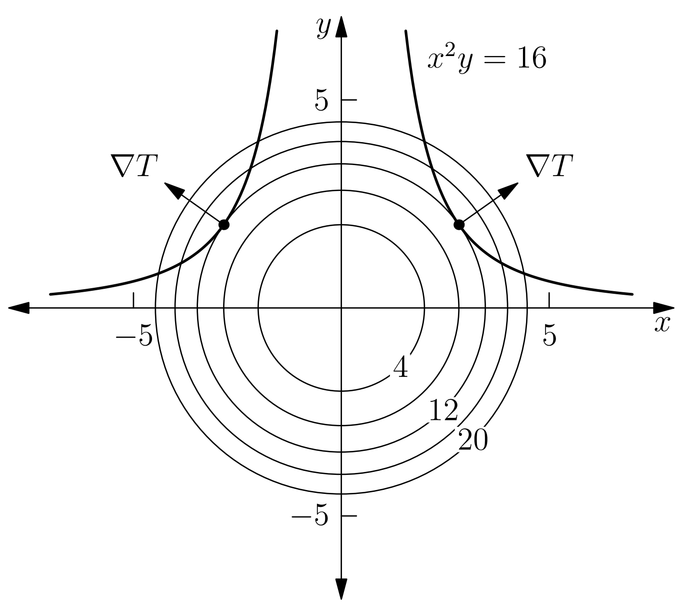
so that $x^2 = 2(2^2) = 8 \implies x = \pm\sqrt{8}$

$\therefore$ the points of extreme temperature on $g(x,y) = 16$ are $(\pm\sqrt{8}, 2)$.

Since $T(\pm\sqrt{8}, 2) = 8 + 4 = 12$, the minimum of T, if there is one, is 12.

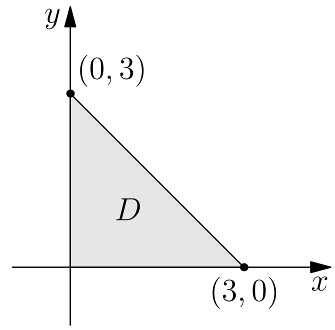
(b) The graph of $x^2y = 16$ is shown below. Sketch some level curves of $T(x,y)$ on this graph. Also show the orientation of ∇T at the point(s) of minimum T on the constraining set. Use this to justify why your answer in (a) truly gives a minimum for T, not a maximum.

/2



All points on $x^2y = 16$ are on level curves of T with $T \geq 12$, so $T(\pm\sqrt{8}, 2) = 12$ is the minimum.

/6 **Problem 5:** Evaluate $\iint_D (x-3y)\,dA$ where D is the triangle in the xy-plane with vertices $(0,0)$, $(3,0)$ and $(0,3)$.



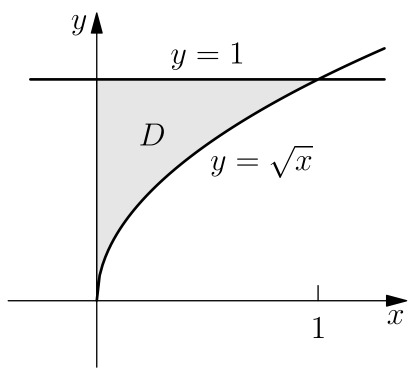
$$\int_0^3 \underbrace{\int_0^{3-x} (x-3y)\,dy}_{\left[xy-\frac{3}{2}y^2\right]_{y=0}^{y=3-x}}\,dx = \int_0^3 \left(x(3-x) - \tfrac{3}{2}(3-x)^2\right) dx$$

$$= \int_0^3 \left(3x - x^2 - \tfrac{3}{2}(3-x)^2\right) dx$$

$$= \tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{2}(3-x)^2\Big|_0^3$$

$$= \left(\tfrac{3}{2}\cdot 9 - \tfrac{1}{3}\cdot 27 + 0\right) - \left(0 - 0 + \tfrac{1}{2}\cdot 27\right) = \boxed{-9}$$

/6 **Problem 6:** Evaluate $\int_0^1 \int_{\sqrt{x}}^1 e^{y^3}\,dy\,dx$



reverse the order of integration: $\int_0^1 \underbrace{\int_0^{y^2} e^{y^3}\,dx}_{\left[xe^{y^3}\right]_{x=0}^{x=y^2}}\,dy = \int_0^1 y^2 e^{y^3}\,dy$

$$= \tfrac{1}{3}e^{y^3}\Big|_0^1 = \boxed{\frac{e-1}{3}}$$