THOMPSON RIVERS UNIVERSITY

MATH 211
Calculus III

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MIDTERM EXAM #1
SOLUTIONS

9 October 2009 09:30–10:20

Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 4 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		7
2		4
3		12
4		4
5		8
TOTAL:		35

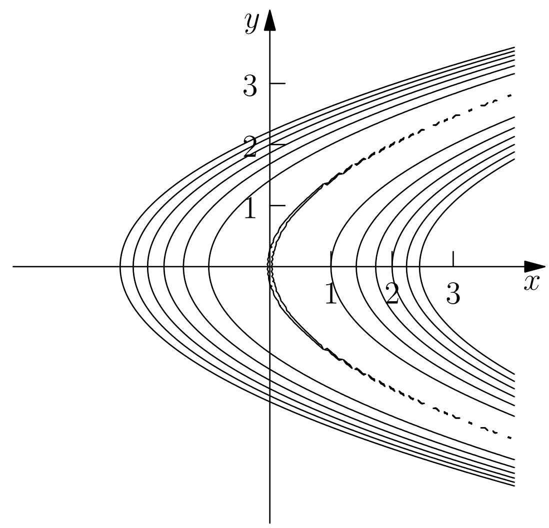
/7

Problem 1: Consider the surface given by the graph of $z = f(x, y)$ where $f(x, y) = (4x - 2y^2)^2$.

(a) Sketch some level curves of this surface. Describe the surface in words.

$$\begin{aligned} z = (4x - 2y^2)^2 = c &\implies 4x - 2y^2 = \pm\sqrt{c} \\ &\implies x = \tfrac{1}{4}(2y^2 \pm \sqrt{c}) \end{aligned}$$

The surface is in the shape of a valley whose bottom lies along the parabola $x = y^2/2$, with sides rising on either side of this parabola.



(b) Find an equation for the plane tangent to this surface at the point $(1, 2, 16)$.

$$f_x(1,2) = 2 \cdot (4x - 2y^2) \cdot 4\Big|_{(1,2)} = -32$$

$$f_y(1,2) = 2 \cdot (4x - 2y^2) \cdot (-4y)\Big|_{(1,2)} = 64$$

so the equation of the tangent plane is

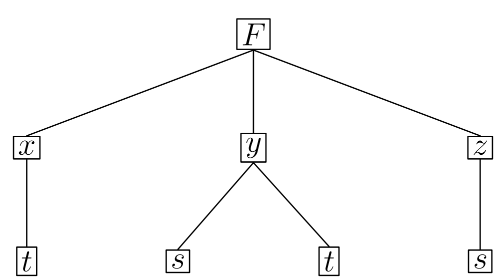
$$\begin{aligned} z &= 16 - 32(x - 1) + 64(y - 2) \\ &= -32x + 64y - 80 \end{aligned}$$

/4

Problem 2: Consider a function $F(x, y, z)$ where $x = x(t)$, $y = y(s, t)$ and $z = z(s)$. Write chain rule expressions for $\dfrac{\partial F}{\partial s}$ and $\dfrac{\partial F}{\partial t}$.

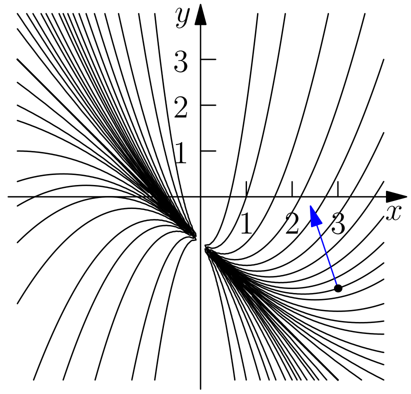
$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\frac{dz}{ds}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$



/12

Problem 3: The temperature T (in °C) at any point in the xy-plane is given by the function $T(x, y) = x^2/(1+x+y)$ (with x and y in cm). Some level curves of T are shown below. An ant is located at the point $(3, -2)$.



(a) In what direction should the ant move if it wishes to decrease its temperature as quickly as possible? Indicate the ant's direction of motion on the graph above.

$$-\nabla T(3,-2) = -\left(\frac{2x\cdot(1+x+y) - 1\cdot x^2}{(1+x+y)^2}, \frac{-x^2}{(1+x+y)^2}\right)\Big|_{(3,-2)} = -\left(\frac{3}{4}, -\frac{9}{4}\right) = \left(-\frac{3}{4}, \frac{9}{4}\right)$$

(b) At what rate does the ant's temperature change if it moves with velocity $\mathbf{v} = (2, -1)$ cm/s?

$$\frac{dT}{dt} = \nabla T \cdot \mathbf{v} = \left(\frac{3}{4}, -\frac{9}{4}\right)\cdot(2,-1) = \frac{15}{4}\ ^\circ\text{C/s (increasing)}$$

(c) Evaluate the directional derivative of T at $(3, -2)$ in the direction of the vector $\mathbf{v} = (-1, 1)$.

form a unit vector in the direction of $\mathbf{v}$: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{(-1,1)}{\sqrt{2}} = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$

then $D_{\mathbf{u}}T = \nabla T\cdot\mathbf{u} = \left(\dfrac{3}{4}, -\dfrac{9}{4}\right)\cdot\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = -\dfrac{3}{\sqrt{2}}$ °C/cm

(d) Evaluate $|\nabla T(3, -2)|$ and describe in words what this quantity means.

$$\Big|T(3,-2)\Big| = \left|\left(\frac{3}{4}, -\frac{9}{4}\right)\right| = \sqrt{\frac{9}{16}+\frac{81}{16}} = \frac{\sqrt{90}}{4}\ ^\circ\text{C/cm}$$

This quantity is the rate of change of temperature in the direction of fastest increase.

/4

Problem 4: Consider the function
$$u(x,t) = \sin(3x)e^{-9kt}$$
where k is a constant. Show that $u_t - ku_{xx} = 0$.

$$u_t = -9k\sin(3x)e^{-9kt} = -9ku$$

$$u_x = 3\cos(3x)e^{-9kt} \implies u_{xx} = -9\sin(3x)e^{-9kt} = -9u$$

$$\implies u_t - ku_{xx} = -9k\sin(3x)e^{-9kt} - k(-9)\sin(3x)e^{-9kt} = -9ku - k(-9u) = 0$$

/8

Problem 5: Let $f(x,y) = \ln(xy)$.

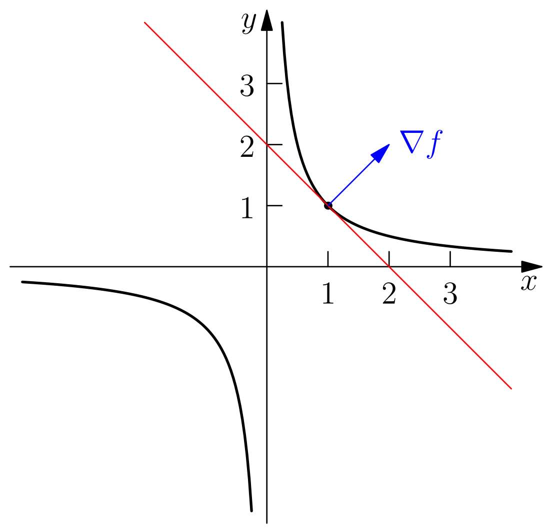
(a) Evaluate $\nabla f(1,1)$ and use it to find an equation for the tangent line to the level curve $f(x,y) = 0$ at the point $(1,1)$. Sketch this curve, the tangent line, and the gradient vector.

/5

$$\nabla f(1,1) = \left(\frac{1}{y}, \frac{1}{x}\right)\Big|_{(1,1)} = \boxed{(1,1)}$$

At $(1,1)$ the normal vector to the level curve is given by
$$\mathbf{n} = \nabla f(1,1) = (1,1)$$
so the equation of the tangent line is
$$\mathbf{x}\cdot\mathbf{n} = \mathbf{x}_0\cdot\mathbf{n} \implies (x,y)\cdot(1,1) = (1,1)\cdot(1,1)$$
$$\implies \boxed{x+y=2}$$

To sketch $f(x,y) = 0$:
$$\ln(xy) = 0 \implies xy = 1 \implies y = \frac{1}{x}$$



(b) Find the domain and range of f. Sketch the domain.

/3

f can be evaluated provided $xy > 0$, so $D(f) = \{(x,y) : xy > 0\}$.

xy can take any value in $\mathbb{R}$ and $R(\ln) = \mathbb{R}$, so $R(f) = \mathbb{R}$.