THOMPSON RIVERS UNIVERSITY

MATH 2110
Calculus 3

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MIDTERM EXAM #2
SOLUTIONS

21 Nov 2019 11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		5
2		6
3		6
4		6
5		6
6		6
TOTAL:		35

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Problem 1: Find the length of the curve

$$\mathbf{r}(t) = a\cos t\,\hat{\mathbf{\imath}} + a\sin t\,\hat{\mathbf{\jmath}} + bt\,\hat{\mathbf{k}}$$

between the points $(a, 0, 0)$ and $(-a, 0, \pi b)$. The numbers a and b are constants.

Note that $\mathbf{r}(0) = (a, 0, 0)$ and $\mathbf{r}(\pi) = (-a, 0, \pi b)$.

We have

$$\mathbf{r}'(t) = (-a\sin t, a\cos t, b) \implies |\mathbf{r}(t)| = \sqrt{a^2\sin^2 + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}$$

and so

$$L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{a^2+b^2}\,dt = \boxed{\pi\sqrt{a^2+b^2}}$$

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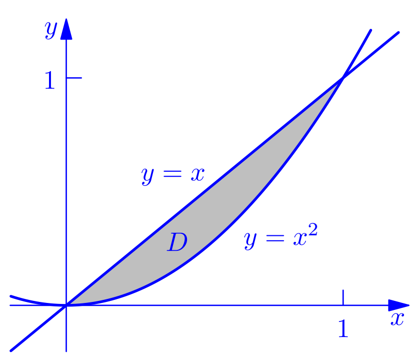
Problem 2: Evaluate $\iint_D (x + y^2)\,dA$ where D is the region of the xy-plane bounded by the graphs of $y = x$ and $y = x^2$.



$$\begin{aligned}\iint_D (x+y^2)\,dA &= \int_0^1 \int_{x^2}^{x} (x+y^2)\,dy\,dx \\ &= \int_0^1 \left[xy + \tfrac{1}{3}y^3\right]_{y=x^2}^{y=x} dx \\ &= \int_0^1 (x^2 + \tfrac{1}{3}x^3) - (x^3 + \tfrac{1}{3}x^6)\,dx \\ &= \int_0^1 (x^2 - \tfrac{2}{3}x^3 - \tfrac{1}{3}x^6)\,dx \\ &= \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 - \tfrac{1}{21}x^7\Big|_0^1 = \frac{1}{3} - \frac{1}{6} - \frac{1}{21} = \boxed{\frac{5}{42}}\end{aligned}$$

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Problem 3: Consider the curve in the xy-plane defined by the equation $xy = 1$. Calculate its radius of curvature at the point $(1, 1, 0)$.

Parametrize the curve with $x = t$, $y = \frac{1}{x} = t^{-1}$ and $z = 0$, i.e. by the vector function

$$\mathbf{r}(t) = (t, t^{-1}, 0).$$

Note that $\mathbf{r}(1) = (1, 1, 0)$ gives the point of interest.

Then we have

$$\mathbf{r}'(t) = (1, -t^{-2}, 0)\mathbf{r}'(1) = (1, -1, 0)$$
$$\mathbf{r}''(t) = (0, 2t^{-3}, 0)\mathbf{r}''(1) = (0, 2, 0).$$

At $t = 1$ we get

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = (0, 0, 2).$$

Thus the curvature at $(1, 1, 0)$ is

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{2}{(\sqrt{2})^3} = \frac{1}{\sqrt{2}}$$

and the radius of curvature is

$$\rho = \frac{1}{\kappa} = \boxed{\sqrt{2}}$$

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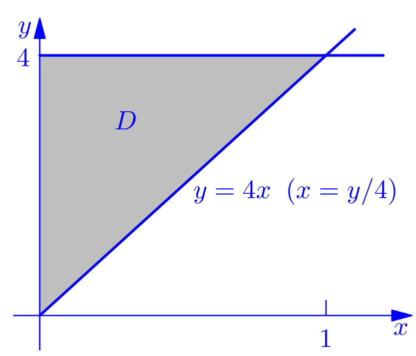
Problem 4: Evaluate by first reversing the order of integration:

$$\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx.$$



$$\begin{aligned}
\int_0^1 \int_{4x}^4 e^{-y^2}\, dy\, dx &= \iint_D e^{-y^2}\, dA \\
&= \int_0^4 \int_0^{y/4} e^{-y^2}\, dx\, dy \\
&= \int_0^4 e^{-y^2} \underbrace{\int_0^{y/4} dx}_{y/4}\, dy \\
&= \int_0^4 \tfrac{1}{4} y e^{-y^2}\, dy \\
&= -\tfrac{1}{8} e^{-y^2} \Big|_0^4 = \boxed{\tfrac{1}{8}(1 - e^{-16})}
\end{aligned}$$

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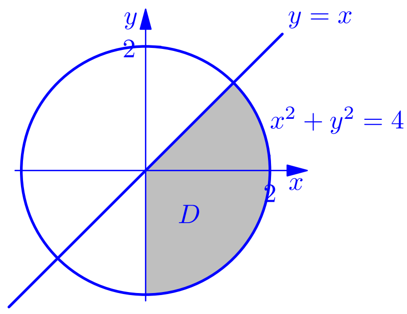
Problem 5: Use polar coordinates to evaluate

$$\iint_D xy\sqrt{x^2+y^2}\,dA$$

where $D = \{(x,y) : x^2+y^2 \le 4,\ y \le x,\ x \ge 0\}$.



$$\begin{aligned}
\iint_D xy\sqrt{x^2+y^2}\,dA &= \int_{-\pi/2}^{\pi/4}\int_0^2 (r\cos\theta)(r\sin\theta)r(r\,dr)\,d\theta \\
&= \underbrace{\int_{-\pi/2}^{\pi/4} \cos\theta\sin\theta d\theta}_{\frac{1}{2}\sin^2\theta\Big|_{-\pi/2}^{\pi/4}} \underbrace{\int_0^2 r^4\,dr}_{\frac{32}{5}} \\
&= \frac{16}{5}\big(\underbrace{\sin\tfrac{\pi}{4}}_{1/\sqrt{2}}\big)^2 - \frac{16}{5}\big(\underbrace{\sin(-\tfrac{\pi}{2})}_{-1}\big)^2 \\
&= \frac{8}{5} - \frac{16}{5} = \boxed{-\frac{8}{5}}
\end{aligned}$$

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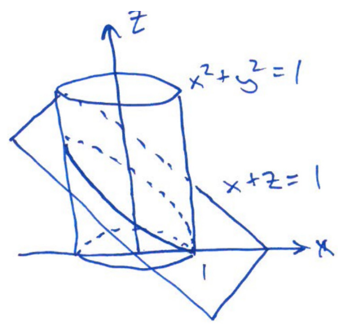
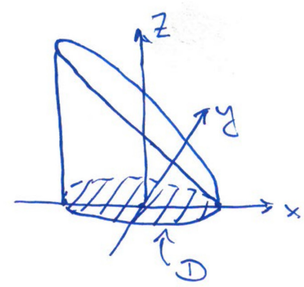
Problem 6: Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$, the plane $z = 0$, and the plane $x + z = 1$.





$x + z = 1 \implies z = 1 - x$. If D is the unit circle in the xy-plane then

$$V = \iint_D (1 - x)\, dA.$$

A clever evaluation of the integral:

$$V = \iint_D (1-x)\, dA = \underbrace{\iint_D dA}_{\text{(area of } D)=\pi} - \underbrace{\iint_D x\, dA}_{0 \text{ by symmetry}} = \boxed{\pi}$$

A more computational solution (using polar coordinates):

$$\begin{aligned} V = \iint_D (1-x)\, dA &= \int_0^{2\pi}\int_0^1 (1 - r\cos\theta) r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 (r - r^2\cos\theta)\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 r\, dr\, d\theta - \int_0^{2\pi}\int_0^1 r^2 \cos\theta\, dr\, d\theta \\ &= \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^1 r\, dr}_{\frac{1}{2}} - \underbrace{\int_0^{2\pi} \cos\theta\, d\theta}_{0} \int_0^1 r^2\, dr = \boxed{\pi} \end{aligned}$$