THOMPSON RIVERS UNIVERSITY

MATH 2110
Calculus 3

Instructor: Richard Taylor

MIDTERM EXAM #1
SOLUTIONS

17 Oct 2019 11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 6 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		4
2		6
3		10
4		8
5		8
6		3
7		5
TOTAL:		44

/4 **Problem 1:** Suppose $u = x^2 - xy$ where $x = s\cos t$ and $y = t\sin s$.

/2 (a) Write a chain rule for $\dfrac{\partial u}{\partial t}$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

/2 (b) Evaluate $\dfrac{\partial u}{\partial s}$ (but do not simplify).

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}$$

$$= \boxed{(2x - y)(\cos t) + (-x)(t\cos s)}$$

.

/6 **Problem 2:** Consider the function $f(x, y) = \sin x + \sin y + \sin(x + y)$.

/4 (a) Find the linear approximation of $f(x, y)$ for (x, y) close to $(0, 0)$.

$$\begin{matrix} f_x = \cos x + \cos(x+y) \\ f_y = \cos y + \cos(x+y) \end{matrix} \implies \begin{matrix} f(0,0) = 0 \\ f_x(0,0) = 2 \\ f_y(0,0) = 2 \end{matrix}$$

The linear approximation is

$$L(x, y) = f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0)$$

$$= 0 + 2(x - 0) + 2(y - 0) = \boxed{2x + 2y}$$

/2 (b) Find an equation for the tangent plane to the graph of $z = f(x, y)$ at the point $(0, 0)$.

This is just the linear approximation found above:

$$\boxed{z = L(x, y) = 2x + 2y}$$

.

/10 **Problem 3:** Let $f(x, y) = x^2 + kxy + y^2$, where k is a constant.

(a) Show that f has a critical point at $(0, 0)$ no matter what value is assigned to k.

/3

We have

$$\begin{array}{l} f_x = 2x + ky \\ f_y = kx + 2y \end{array} \implies f_x(0,0) = f_y(0,0) = 0$$

so f has a critical point at $(0, 0)$, regardless of the value of k.

(b) For what value(s) of k will f have a saddle point at $(0, 0)$?

/3

Apply the second derivative test:

$$\begin{array}{l} f_{xx} = 2 \\ f_{yy} = 2 \\ f_{xy} = k \end{array} \implies D = f_{xx}f_{yy} - [f_{xy}]^2 = 4 - k^2$$

So $(0, 0)$ will be a saddle point if

$$4 - k^2 < 0 \implies k^2 > 4 \implies \boxed{|k| > 2 \quad (k < -2 \text{ or } k > 2)}$$

(c) For what value(s) of k will f have a local maximum at $(0, 0)$?

/2

A local max at $(0, 0)$ requires $f_{xx} = 2 < 0$ which gives a contradiction. There are no values of k for which f has a local min at $(0, 0)$.

(d) For what value(s) of k will f have a local minimum at $(0, 0)$?

/2

For a local min at $(0, 0)$ we require $D > 0$ and $f_{xx} > 0$:

$$\left\{ \begin{array}{l} 4 - k^2 > 0 \\ 2 > 0 \end{array} \right. \implies k^2 < 4 \implies \boxed{-2 < k < 2}$$

/8

Problem 4: Use the method of Lagrange multipliers to find the minimum distance from the point $(0,1)$ to the parabola $y = x^2$.

Let (x, y) be a point on the parabola. It is easier to minimize the *square* of the distance to $(0, 1)$, i.e. the function

$$f(x,y) = x^2 + (y-1)^2.$$

We need to minimize $f(x, y)$ subject to the constraint

$$\underbrace{y - x^2}_{g(x,y)} = 0.$$

The method of Lagrange multipliers gives

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases} \implies \begin{cases} 2x = \lambda(-2x) \\ 2(y-1) = \lambda(1) \\ y = x^2. \end{cases}$$

The first equation gives

$$2x(1+\lambda) = 0 \implies x = 0 \text{ or } \lambda = -1.$$

<u>case $x = 0$:</u>

$$y = 0^2 = 0 \implies (x,y) = (0,0).$$

<u>case $\lambda = -1$:</u>

$$2(y-1) = -1 \implies y = \frac{1}{2}$$

$$x^2 = \frac{1}{2} \implies x = \pm\frac{1}{\sqrt{2}} \implies (x,y) = \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

We have:

$$f(0,0) = 1 \quad \text{(max)}$$

$$f\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(\pm\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} \quad \text{(min)}$$

so the minimum distance is

$$\sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}$$

(the point $(0, 0)$ has distance 1, corresponding to a local max of f.)

/8 **Problem 5:** In a certain region of the xy-plane, the temperature T [in °C] varies according to the function

$$T(x, y) = 48 - \tfrac{4}{3}x^3 - 3y^2$$

where x, y are measured in cm.

(a) At the point $(1, -1)$ find the direction of the most rapid temperature decrease.

/2

$$\nabla T = (-4x^2, -6y) \implies \boxed{-\nabla T(1, -1) = (4, -6) = 4\hat{\mathbf{i}} - 6\hat{\mathbf{j}}}$$

(b) Find the rate of temperature change at the point $(1, -1)$ in the direction of most rapid increase.

/2

$$|\nabla T(1, -1)| = |(-4, 6)| = \sqrt{4^2 + 6^2} = \boxed{\sqrt{52}\,^\circ\text{C/cm} = 2\sqrt{13}\,^\circ\text{C/cm}}$$

(c) Find the rate of temperature change at the point $(1, -1)$ in the direction away from the origin.

/2

We want the directional derivative of T in the direction of $\mathbf{v} = (1, -1)$, which corresponds to the unit vector

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(1, -1)}{\sqrt{2}}.$$

Thus

$$D_{\mathbf{u}}T(1, -1) = \nabla T \cdot \mathbf{u} = (-4, 6) \cdot \frac{(1, -1)}{\sqrt{2}} = \boxed{-\frac{10}{\sqrt{2}}\,^\circ\text{C/cm}}$$

(d) An ant at the point $(1, -1)$ moves with velocity vector $\mathbf{v} = (3\hat{\mathbf{i}} + 5\hat{\mathbf{j}})$ cm/s. What rate of temperature change does it experience?

/2

$$\frac{dT}{dt} = \nabla T \cdot \mathbf{v} = (-4, 6) \cdot (3, 5) = \boxed{18\,^\circ\text{C/s}}$$

/3

Problem 6: Suppose the curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect at right angles at a point P. What condition must be satisfied by the partial derivatives of f and g at P?

Since the curves intersect at right angles, so do their normal vectors. These are given by ∇f and ∇g, so

$$\nabla f \cdot \nabla g = 0 = (f_x, f_y) \cdot (g_x, g_y)$$
$$\implies \boxed{f_x g_x = -f_y g_y}$$

/5

Problem 7: Find an equation for the tangent plane to the graph of $x^2y^2 + 2x + z^3 = 16$ at the point $(2, 1, 2)$.

Let $f(x, y, z) = x^2y^2 + 2x + z^3$, then the surface in question is a level surface of f so its normal at $(2, 1, 2)$ is

$$\nabla f(2, 1, 2) = (2xy^2 + 2, 2x^2y, 3z^2)\Big|_{(2,1,2)} = (6, 8, 12).$$

The "normal form" for the equation of the plane with this normal, through the point $(2, 1, 2)$ is:

$$[(x, y, z) - (2, 1, 2)] \cdot (6, 8, 12) = 0$$
$$\implies 6(x - 2) + 8(y - 1) + 12(z - 2) = 0$$
$$\implies 6x + 8y + 12z = 44$$
$$\implies \boxed{3x + 4y + 6z = 22}$$