THOMPSON RIVERS UNIVERSITY

# Calculus II

Lecture Notes for MATH 124

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This is not a textbook. Please, buy the required textbook for the course and use it. My aim is to produce a condensed set of notes to help me organize and improve upon my lecture material. Thus, very few of my examples include full solutions; these are to be worked out in class, along with the sundry other details, diagrams, humorous anecdotes, and everything else that makes attending lectures worthwhile. Do not use these as a substitute for taking notes in class.

# Calculus II
Lecture Notes for MATH 124

## Course Syllabus

Section numbers refer to the assigned text: James Stewart, *Single Variable Calculus: Concepts and Contexts*, 3rd edition, Thomson, 2006.

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# 1 Preliminaries

Lec. #1

Central problem of Differential Calculus:

given $f(x)$, find slope of $y = f(x)$ at $x = a$.

- this led to definition of the *derivative*, which in turn had many applications

Central problem of Integral Calculus:

given $f(x)$, find area under $y = f(x)$ from $x = a$ to $x = b$.

- similarly, this will lead to definition of the *integral*, which will also have many applications

- the two central problems turn out to be inverses of each other; this gives the Fundamental Theorem of Calculus, which we will study in detail

# 2 Areas Under Graphs

To computing the area under a graph, we start with an approximation.

**Example 2.1.** Find the area under $y = x^2$ between $x = 0$ and $x = 1$.

If I gave you this shape as a cutout piece of paper and asked to calculate the area, you might cut it up into rectangles and add the areas. This is what we'll do.

Cutting into 4 vertical rectangles, with heights given by the height of the graph at their *left* endpoints, we get an area

$$S_4 = 0 + (0.25)(0.25)^2 + (0.25)(0.5)^2 + (0.25)(0.75)^2 \approx 0.219,$$

which is an *underestimate* of the true area.

We could also evaluate heights at *right* endpoints, and get

$$S_4 = (0.25)(0.25)^2 + (0.25)(0.5)^2 + (0.25)(0.75)^2 + (0.25)(1)^2 \approx 0.469$$

which is an *overestimate* of the true area.

*Key Idea*: we can get a *better* approximation if we use more, narrower rectangles (again using right endpoints):

$$S_{10} = (0.1)(0.1)^2 + (0.1)(0.2)^2 + (0.1)(0.3)^2 + \cdots + (0.1)(0.9)^2 + (0.1)(1)^2 = 0.385$$
$$S_{50} = (0.02)(0.02)^2 + (0.02)(0.04)^2 + \cdots + (0.02)(0.98)^2 + (0.02)(1)^2 = 0.3434$$
$$S_{100} = (0.01)(0.01)^2 + (0.01)(0.02)^2 + \cdots + (0.01)(0.99)^2 + (0.01)(1)^2 = 0.33835$$

Lec. #2

Note that the areas get smaller: each overestimates the true area by less.

What we'd really like is to use an *infinite* number of rectangles to get the *exact* area, because as $n \to \infty$ the error tends to zero. But this gives nonsense: an infinite number of rectangles with zero width gives... zero area!? Rather, the calculus way to do this is to say that

$$A = \lim_{n\to\infty} S_n.$$

Our goal is to calculate this limit. To do this we need a formula for $S_n$, the area that results if we use $n$ rectangles.

Note that each rectangle with have a width $\Delta x = 1/n$.

The height of the $i$'th rectangle comes from evaluating $f(x)$ at the corresponding $x$-coordinate:

$$x_1 = \Delta x, \quad x_2 = 2\Delta x, \quad x_3 = 3\Delta x, \quad \ldots$$

so in general,

$$x_i = i\Delta x = i\left(\frac{1}{n}\right) = \frac{i}{n}.$$

So the *height* of the $i$'th rectangle is

$$f(x_i) = \left(\frac{i}{n}\right)^2 = \frac{i^2}{n^2}.$$

Now the *area* of the $i$'th rectangle is

$$A_i = \text{height} \times \text{width} = f(x_i)\Delta x = \left(\frac{i^2}{n^2}\right)\frac{1}{n} = \frac{i^2}{n^3}$$

So, finally, the *total* area of all $n$ rectangles is

$$\begin{aligned} S_n &= A_1 + A_2 + A_3 + \cdots + A_n \\ &= \sum_{i=1}^{n} A_i \\ &= \sum_{i=1}^{n} \frac{i^2}{n^3} \quad \text{(...aside on “sigma notation”)} \\ &= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\ &= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(2n+1)}{6n^3}. \end{aligned}$$

The hard work is done... now we can use this formula to easily compute the area for *any* value of $n$. For example,

$$\begin{aligned} S_{100} &= \frac{100 \times 101 \times 201}{6 \times 100^3} = 0.33835 \\ S_{1000} &= \frac{1000 \times 1001 \times 2001}{6 \times 1000^3} = 0.3338335 \\ S_{10000} &= \frac{10000 \times 10001 \times 20001}{6 \times 10000^3} = 0.333383335 \end{aligned}$$

Even better, though, we can now evaluate $\lim_{n\to\infty}$ to get the *exact* area. We suspect this limit is 1/3, and we can prove it: :

$$\begin{aligned} A = \lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\ &= \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} \\ &= \lim_{n\to\infty} \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{6} \\ &= \frac{2+0+0}{6} = \frac{1}{3}. \end{aligned}$$

**Aside on sigma notation and useful formulas**

$\sum_{i=1}^{n} A_i$ is a shorthand for $A_1 + A_2 + \cdots + A_n$. In general, anything of the form $\sum_{i=1}^{n}(\cdots)$ means "plug all values of $i$ from 1 to $n$ in the formula $(\cdots)$, and sum all $n$ terms that result."

In doing area calculations, there are some useful formulas to have:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

and of course

$$\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}} = n.$$

It is also useful to rearrange sums using the following:

$$\sum_{i=1}^{n} C a_i = C \sum_{i=1}^{n} a_i$$
$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

These allow us to do sums like

$$\sum_{i=1}^{n} \left(3i - i^2\right) = \sum_{i=1}^{n} 3i - \sum_{i=1}^{n} i^2 = 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2 = 3\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}.$$

Let's do another, simpler area that we already know the answer for:

**Example 2.2.** Find the area under $y = 2x$ from $x = 0$ to $x = 3$.

(note: since the region is a triangle, we know the area is $\frac{1}{2} \times 3 \times 6 = 9$.)

Now $\Delta x = \frac{3}{n}$, and $x_i = i\Delta x = \frac{3i}{n}$, and we get

$$\begin{aligned} S_n = \sum_{i=1}^{n} A_i &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \sum_{i=1}^{n} \left(2 \cdot \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\ &= \sum_{i=1}^{n} \frac{18i}{n^2} \quad \text{and now we evaluate the sum...} \\ &= \frac{18}{n^2} \sum_{i=1}^{n} i \\ &= \frac{18}{n^2} \frac{n(n+1)}{2}. \end{aligned}$$

So the exact area is

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{9n(n+1)}{n^2} = \lim_{n\to\infty} \frac{9n^2 + 9n}{n^2} = \lim_{n\to\infty} \left(9 + \frac{9}{n}\right) = 9.$$

Lec. #3

**Example 2.3.** Find the area under $y = x$ between $x = 2$ and $x = 4$.

**Example 2.4.** Find the area under $y = x^3$ between $x = 0$ and $x = 1$.

# 3 The Definite Integral

To find area under $y = f(x)$ between $x = a$ and $x = b$, we started with the area of $n$ vertical strips:

$$S_n = \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where} \quad \Delta x = \frac{b-a}{n},$$

then we found the exact area by letting $n \to \infty$:

$$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[ \sum_{i=1}^{n} f(x_i)\Delta x \right].$$

This complicated expression occurs so often that there is a shorthand notation:

**Definition 3.1.**

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \left[ \sum_{i=1}^{n} f(x_i)\Delta x \right]$$

is called the **definite integral** of $f$ from $a$ to $b$.

It can be interpreted as the area under $y = f(x)$ between $x = a$ and $x = b$, but it will also have other interpretations (as was the case for the derivative).

So, for example, $\int_0^\pi \sin(x)\, dx$ is a number, which we can interpret as the area under $y = \sin x$ between $x = 0$ and $x = \pi$.

You can think of $dx$ as an infinitesimal $\Delta x$. The symbol $\int_a^b$ means "sum" from $a$ to $b$, in the sense of the sum of an infinite number of rectangles each with infinitesimal area $dA = f(x)\, dx$ (height times infinitestimal width). (Later, this will be a very useful way of thinking about integrals.) As $n \to \infty$ (and $\Delta x \to 0$),

$$\sum_{i=1}^{n} f(x)\,\Delta x \quad \text{becomes} \quad \int_a^b f(x)\, dx.$$

Note the correspondence between the symbols in the two expressions.

The $dx$ is also notational: it tells us that $x$ is the variable over which the integral is calculated. So, for example, $\int_0^\pi \sin(x)\, dx$ and $\int_0^\pi \sin(t)\, dt$ both represent the same number, the same area: it doesn't matter what *label* we use for the horizontal axis. So the variables $x$ and $t$ are somewhat arbitrary... we call such variables *dummy variables*.

In the same way, the index $i$ in the sum $\sum_{i=1}^{n} f(x_i)\Delta x$ is a dummy variable: relabelling the index and writing e.g. $\sum_{k=1}^{n} f(x_k)\Delta x$ doesn't change the meaning or value of the expression. The name of the index variable is arbitrary.

# 4 Properties of the Definite Integral

1. if $f(x) \geq 0$ then $\int_a^b f(x)\, dx$ is the area under the graph between $x = a$ and $x = b$.
2. otherwise, $\int_a^b f(x)\, dx =$ (area above $x$-axis) − (area below $x$-axis).

3. $\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$

4. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

5. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

**Example 4.1.** Using 3. we can calculate:

$$\int_0^1 6x^2 = 6\int_0^1 x^2 = 6\left(\frac{1}{3}\right) = 2.$$

**Example 4.2.** Using 3. and 4. we can calculate:

$$\int_0^1 (2+3x-5x^2)\,dx = \underbrace{\int_0^1 2\,dx}_{=2 \text{ (area of rectangle)}} + 3\underbrace{\int_0^1 x\,dx}_{=\frac{1}{2}\text{(area of triangle)}} - 5\underbrace{\int_0^1 x^2\,dx}_{=\frac{1}{3}\text{(from last day))}}$$
$$= 2 + \frac{3}{2} - \frac{5}{3} = \frac{11}{6}.$$

Lec. #4

**Example 4.3.** To evaluate $\int_0^3 (1-x)\,dx$ we could keep track of positive and negative contributions from different areas:

$$\int_0^3 (1-x)\,dx = \underbrace{\int_0^1 (1-x)\,dx}_{=\frac{1}{2}\text{ (triangle)}} + \underbrace{\int_1^3 (1-x)\,dx}_{=-2\text{ (triangle)}} = -\frac{3}{2}$$

*or* we could use the properties of integrals:

$$\int_0^3 (1-x)\,dx = \underbrace{\int_0^3 1\,dx}_{=3\text{ (rectangle)}} - \underbrace{\int_0^3 x\,dx}_{=\frac{9}{2}\text{ (triangle)}} = -\frac{3}{2}.$$

**Example 4.4.** Evaluate $\int_1^5 (4-2x)\,dx$.

**Example 4.5.** Evaluate $\int_{-1}^1 \sqrt{1-x^2}\,dx$.

### 4.1 Integrals and symmetry

**Example 4.6.** Use symmetry to evaluate: (a) $\displaystyle\int_{-2}^2 x^3 dx$ (b) $\displaystyle\int_{-\pi}^{\pi} \sin x\,dx$ (c) $\displaystyle\int_0^{2\pi} \sin x\,dx$

## 5 Fundamental Theorem of Calculus (Part I)

Calculating integrals (areas) using the definition 3.1 is hard—even impossible for many cases. (This is analogous to calculating derivatives from the definition, i.e. $\lim_{h\to 0}\cdots$ etc.) We need a better way to evaluate integrals. The main tool is the following theorem.

**Theorem 5.1** (Fundamental Theorem of Calculus, Part I)**.** *If $f$ is continuous on the interval $[a,b]$ then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

*where $F$ is a function such that $F'(x) = f(x)$ (i.e. $F$ is an* antiderivative *of $f$).*

This theorem says that calculating an integral is basically equivalent to finding an antiderivative. So integration is like the opposite, or inverse, of differentiation. (Later, in Part II of the theorem, we will see that this is true in an even broader sense.)

**Example 5.1.** We can use the Fundamental Theorem to evaluate, e.g., $\int_0^1 x^2\,dx$. Here $f(x) = x^2$. We need an antiderivative $F(x)$ so that $F'(x) = x^2$. The function $F(x) = \frac{1}{3}x^3$ works, so:

$$\int_0^1 x^2\,dx = F(1) - F(0) = \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 = \frac{1}{3}.$$

We will be evaluating things like $F(b) - F(a)$ a lot, so the following shorthand is useful:

$$F(x)\Big|_a^b = F(b) - F(a).$$

In the previous example we could have written $\displaystyle\int_0^1 x^2\,dx = \left(\frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{3}.$

**Example 5.2.** Evaluate: (a) $\displaystyle\int_0^2 x\,dx$ (b) $\displaystyle\int_0^1 x^3\,dx$ (c) $\displaystyle\int_0^\pi \sin x\,dx$ (d) $\displaystyle\int_0^9 \sqrt{x}\,dx$

## 5.1 Net Change Theorem

Lec. #5

There is another very useful interpretation of the derivative, which comes from the fundamental theorem. Recall that if the function $x(t)$ gives the *position* or *displacement* of an object at time $t$, then the derivative $x'(t)$ gives the *velocity* at time $t$. Notice that $x(t)$ is an antiderivate of $x'(t)$. So the fundamental theorem says:

$$\int_a^b x'(t)\,dt = x(b) - x(a).$$

Turning this around, it says that the net displacement from time $a$ to time $b$ (i.e. $x(b) - x(a)$) can be found by integrating the velocity function time $t = a$ to $t = b$. This makes physical sense: $x'(t)\,dt$ represent the infinitesimal distance (= velocity × time) covered in the infinitesimal time increment $dt$. The integral sums all these infinitesimal distances over the whole time interval $[a, b]$ to give the total distance.

**Example 5.3.** A particle is travelling along the $x$-axis such that it's velocity at time $t$ is $v(t) = t^2$. How far does the particle travel from time 0 to time 5?

The result above says that the total distance can be found using

$$x(5) - x(0) = \int_0^5 v(t)\,dt = \int_0^5 t^2\,dt = \left(\frac{1}{3}t^3\right)\Big|_0^5 = \frac{125}{3}.$$

**Example 5.4.** A particle is travelling along the $x$-axis such that it's velocity at time $t$ is $v(t) = 4 - t$. How far does the particle travel from time 0 to time 5? Interpret using a graph of $v'(t)$ vs. $t$.

In general, if $f'(t)$ represents the instantaneous rate of change of a quantity $f(t)$, then the fundamental theorem can be expressed as:

**Theorem 5.2** (Net Change Theorem)**.**

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

This allows us to calculate the net change, $f(b) - f(a)$, given only the rate of change, $f'(t)$.

Examples:

- if $f'(t)$ gives the flow rate ($\mathrm{m}^3$/min) of water out of a pipe, then $\int_a^b f'(t)\,dt$ gives the total amount of water ($\mathrm{m}^3$) that comes out of the pipe between time $t = a$ and $t = b$.
- if $f'(t)$ gives the rate at which you use electricity in your home ($ per day), then $\int_a^b f'(t)\,dt$ gives the total amount (in $) that you spend on electricity from time $a$ to time $b$.
- if $f'(t)$ is the rate of population growth in a certain country (# of individuals per year), then $\int_a^b f'(t)\,dt$ gives the total increase in population from time $a$ to time $b$.

## 5.2 The Indefinite Integral

Since the business of computing integrals boils down to finding antiderivatives, it is useful to have a notation for the antiderivative of a function.

**Definition 5.1.** $\displaystyle\int f(x)\,dx$ means "the antiderivative of $f(x)$". That is,

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x).$$

So, for example, $\int 3x^2\,dx = x^3$. But notice that $F(x) = x^3 + 5$ is *also* an antiderivative for $f(x) = 3x^2$.

In fact we could add *any* constant to $F$ and still have an antiderivative for $f$. The *most general* antiderivative for $f(x) = 3x^2$ is $F(x) = x^3 + C$ where $C$ is an arbitrary constant (the so-called "constant of integration").

When we write antiderivatives, we *always* want to use the most general antiderivative that includes the constant of integration. So, for example,

$$\int x\,dx = \frac{1}{2}x^2 + C$$
$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \sin x\,dx = \cos x + C$$
$$\int \cos x\,dx = -\sin x + C$$

**Example 5.5.** Evaluate: (a) $\displaystyle\frac{1}{1+x^2}\,dx$ (b) $\displaystyle\int (1+x)(2-x)\,dx$ (c) $\displaystyle\int \frac{1+x}{x^3}\,dx$ (d) $\displaystyle\int e^{3x}\,dx$

One special antiderivative is

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Without the absolute value, the integral of $1/x$ would only be defined for $x > 0$ because of the $\ln x$ is undefined for $x \le 0$. But there must be some function, defined for $x < 0$, whose derivative is $1/x$. You can check that $\ln(-x)$ is such a function. So we have

$$\int \frac{1}{x}\,dx = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

Of course the piecewise function on the righthand side can be written simply as $\ln|x|$, and we also must remember to add a constant of integration.

## 6 Substitution Rule

Lec. #6

What if we need to calculate an integral, but we can't easily recognize an antiderivative? e.g.,

$$\int 3x^2 e^{x^3} = ?$$

One useful trick involves a *change of variables*. We will rewrite the integral in terms of a new variable $u$, related to $x$ by

$$u = x^3.$$

To integrate with respect to $u$, we need to relate $du$ (the differential of $u$) to $dx$ (the differential of $x$. Differentiating the equation above we get

$$du = 3x^2 dx.$$

Notice that the expression $3x^2\,dx$ occurs in the integral, so we can replace this with $du$ to get:

$$\int 3x^2 e^{x^3} = \int e^u\,du = e^u + C = e^{x^3} + C.$$

Note how we revert to writing the integral in terms of $x$; the variable $u$ was used only as an intermediate step. Check that $e^{x^3}$ actually is an antiderivative for $3x^2 e^{x^3}$. Notice that this worked because the derivative of $x^3$ also appears in the integral.

In general, if $u = g(x)$ is differentiable then

$$\int f\big(g(x)\big)g'(x)\,dx = \int f(u)\,du.$$

**Example 6.1.** Evaluate: (a) $\int (3x+5)^8\,dx$ (b) $\int \frac{3x}{\sqrt{1+x^2}}\,dx$ (c) $\int \tan x\,dx$ (d) $\int \frac{\ln x}{5x}\,dx$ (e) $\int \frac{1}{x\ln x}\,dx$ (f) $\int \frac{e^x}{1+e^x}\,dx$ (g) $\int \cos(x)e^{\sin(2x)}\,dx$

Suppose we need to evaluate a *definite* integral using a substitution, e.g.

$$\int_0^1 2x\sqrt{1+x^2}\,dx.$$

There are two methods for doing this, using the subsitution $u = 1 + x^2$, $du = 2x\,dx$. One is to first find an antiderivative:

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} = \frac{2}{3}(1+x^2)^{3/2}$$

then evaluate at the endpoints:

$$\int_0^1 2x\sqrt{1+x^2}\,dx = \frac{2}{3}(1+x^2)^{3/2}\Big|_0^1 = \frac{2}{3}2^{3/2} - \frac{2}{3}.$$

The other method is to change the limits of integration when we change variables. When $x = 0$, $u = 1 + (0)^2 = 1$. When $x = 1$, $u = 1 + (1)^2 = 2$. So

$$\int_0^1 2x\sqrt{1+x^2}\,dx = \int_1^2 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_1^2 = \frac{2}{3}2^{3/2} - \frac{2}{3}.$$

In the second method, we didn't need to rewrite the integral in terms of $x$; instead, we evaluated the definite integral in terms of $u$, take care to change the limits when we changed the variable.

**Example 6.2.** Evaluate: (a) $\int_0^7 \sqrt{4+3x}\,dx$ (b) $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$ (c) $\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

# 7 Areas Between Graphs

Between $y = f(x)$ and $y = g(x)$ (with $f(x) \geq g(x)$) the area of an infinitesimal vertical strip is

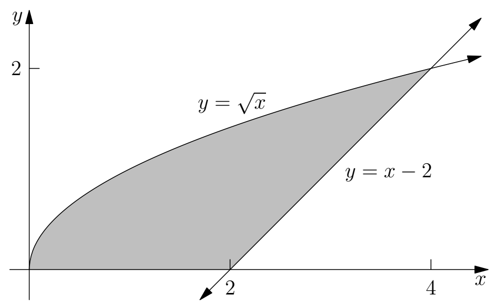
$$dA = \big(f(x) - g(x)\big)\,dx$$

so the area enclosed between $x = a$ and $x = b$ is

$$A = \int dA = \int_a^b \big(f(x) - g(x)\big)\,dx.$$

**Example 7.1.** Find the area enclosed between the curves $y = 2 - x^2$ and $y = -x$

**Example 7.2.** Find the area above the $x$-axis enclosed by $y = \sqrt{x}$, $y = x - 2$ and $x = 0$.




## 7.1 Integration with respect to $y$

Sometimes areas are easier to find by taking *horizontal* strips. In the last example, we can express the bounding curves as

$$x = y + 2 \quad \text{and} \quad x = y^2.$$

An infinitesimal horizontal strip with height $dy$ has area

$$dA = \big((y + 2) - y^2\big)\,dy$$

So the area enclosed is

$$A = \int_0^2 y + 2 - y^2\,dy = \left(\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right)\Big|_0^2 = \frac{10}{3}.$$

# 8 Integration by Parts

Lec. #7

Subsitutions for finding antiderivatives essentially carry out the inverse of the chain rule. It helps find antiderivatives, but not all integrals can be done this way.

Integration by parts is like the inverse of the product rule. Recall that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Integrating both sides gives

$$f(x)g(x) = \int \left[f'(x)g(x) + f(x)g'(x)\right]\,dx.$$

If we let $u = f(x)$ and $v = g(x)$ (so that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$) we can rewrite this as

$$uv = \int v\,du + \int u\,dv.$$

Rearranging gives the formula

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

To apply the formula to a given integral, you need to choose two substitutions: one for $u$, the other for $dv$.

**Example 8.1.** Find $\displaystyle\int x\sin x\,dx$.

Choose $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. The integration by parts formula gives

$$\begin{aligned}\int x\sin x\,dx &= x(-\cos x) - \int(-\cos x)\,dx\\ &= -x\cos x + \int \cos x\,dx\\ &= -x\cos x + \sin x + C.\end{aligned}$$

**Example 8.2.** Find: (a) $\displaystyle\int x\ln x\,dx$ (b) $\displaystyle\int \ln x\,dx$ (c) $\displaystyle\int xe^x\,dx$ (d) $\displaystyle\int x^2e^x\,dx$ (e) $\displaystyle\int \arctan x\,dx$

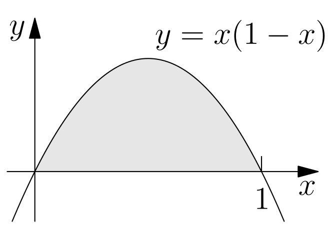
**Example 8.3.** Find $\displaystyle\int e^x\cos x\,dx$.

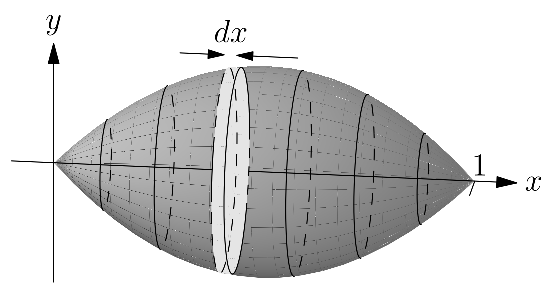
Lec. #8

# 9 Volumes

## 9.1 Method of discs

**Example 9.1.** The region bounded by $y = x(1-x)$ and $y = 0$ is revolved about the $x$-axis. Find the volume of the resulting solid of revolution.




Taking thin cross-sections perpendicular to the $x$-axis, we get circular discs of radius $r = x(1-x)$. Each disc has volume

$$dV = \pi r^2 dx = \pi[x(1-x)]^2\,dx.$$

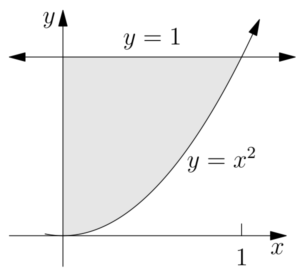
We get the total volume by adding the infinitesimal volume contributions from each disc:

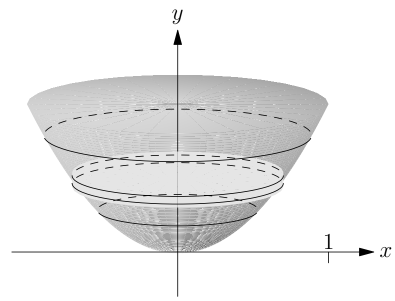
$$V = \int dV = \int_0^1 \pi[x(1-x)]^2\,dx.$$

Evaluate the integral as usual:

$$V = \int_0^1 \pi[x(1-x)]^2\,dx = \pi\int_0^1 (x^2 - 2x^2 + x^4)\,dx = \pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5\right)\Big|_0^1 = \frac{\pi}{30}.$$

**Example 9.2.** The region bounded by $y = x^2$, $y = 1$ and $x = 0$ is revolved about the $y$-axis. Find the volume of revolution.



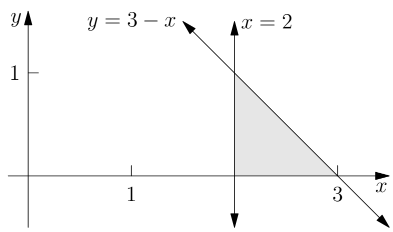

Taking thin cross-sections perpendicular to the $y$-axis we get thin discs of volume

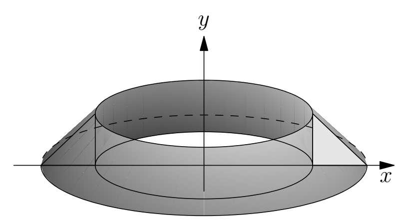
$$dV = \pi r^2 \, dy$$

where $r = x = \sqrt{y}$. The total volume is

$$V = \int dV = \int_0^1 \pi(\sqrt{y})^2 \, dy = \pi \int_0^1 y \, dy = \pi \left(\tfrac{1}{2}y^2\right)\Big|_0^1 = \frac{\pi}{2}.$$

**Example 9.3.** The region bounded by $y = 3 - x$, $y = 0$ and $x = 2$ is revolved about the $y$-axis. Find the volume of revolution.

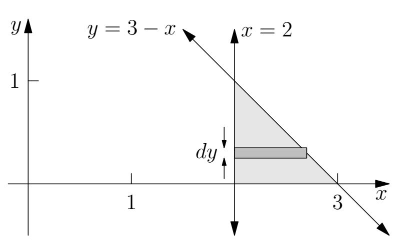


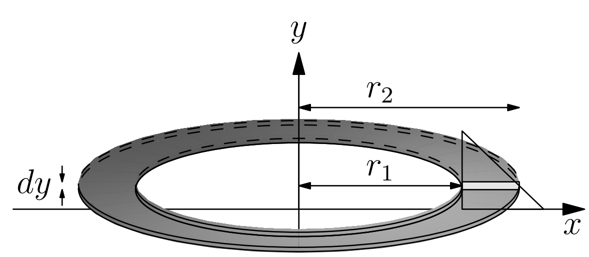




Taking thin cross-sections perpendicular to the $y$-axis we get thin "washers" of volume

$$dV = (\pi r_2^2 - \pi r_1^2) \, dy = \pi(r_2^2 - r_1^2) \, dy$$

where $r_2 = 3 - y$ and $r_1 = 2$. (Imagine taking thin horizontal strips of the bounded region, and revolving each strip about the $y$-axis to form $dV$.)




The total volume is

$$V = \int dV = \int_0^1 \left(\pi(3-y)^2 - \pi(2)^2\right) \, dy = \pi \int_0^1 \left((3-y)^2 - 4\right) \, dy.$$

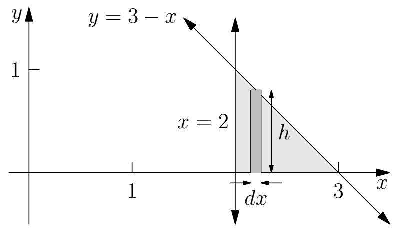
Evaluating the integral we get:

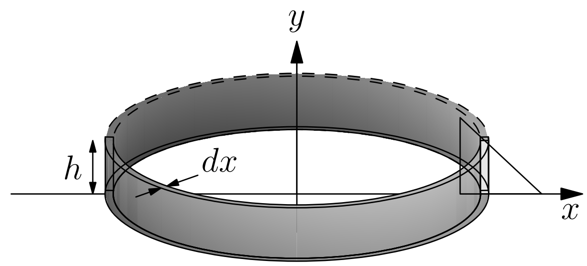
$$V = \pi \int_0^1 \left((3-y)^2 - 4\right) \, dy = \pi \int_0^1 (5 - 6y + y^2) \, dy = \pi \left(5y - 3y^2 + \tfrac{1}{3}y^3\right)\Big|_0^1 = \frac{7\pi}{3}.$$

Lec. #9

## 9.2 Method of shells

**Example 9.4.** The previous example can be solved by taking thin *vertical* strips of the bounded region, and revolving each strip about the $y$-axis to form $dV$.




The infinitesimal volume elements are thin-walled cylinders of volume

$$dV = 2\pi r h dx$$

where $r = x$ and $h = 3 - x$. The total volume is then

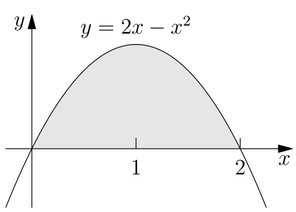
$$V = \int dV = \int_2^3 2\pi x(3-x)\,dx.$$

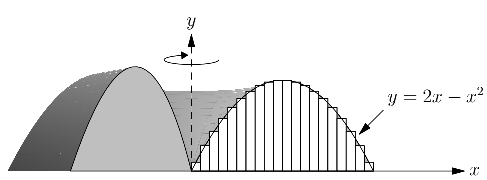
Evaluating the integral we get

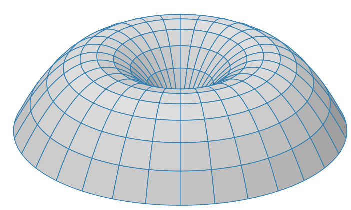
$$V = \int_2^3 2\pi x(3-x)\,dx = \pi \int_2^3 (3x - x^2)\,dx = 2\pi\left(\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_2^3 = 2\pi\left(\frac{9}{2} - \frac{10}{3}\right) = \frac{7\pi}{3}$$

which agrees with the answer we got by the method of discs.

**Example 9.5.** The region bounded by $y = 2x - x^2$ and $y = 0$ is revolved about the $y$-axis. Find the volume of revolution.



Using the method of shells we get thin-walled cylinders of volume

$$dV = 2\pi r h dx = 2\pi x(2x - x^2)dx$$

so the total volume is

$$V = \int dV = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx = 2\pi\left(\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right)\Big|_0^2 = \frac{8\pi}{3}.$$

# 10 Fundamental Theorem of Calculus (Part II)

Lec. #10

The first part of the Fundamental Theorem says that integration is effectively the inverse of differentiation, in that

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b.$$

That is, if you differentiate a function $f$, then integrate, you get back to $f$. The second part of the theorem says the opposite is also true: if you integrate $f$ then differentiate, you also get back to $f$.

**Theorem 10.1.** *If $f$ is continuous on $[a,b]$ then the function*

$$g(x) = \int_a^x f(s)\,ds \quad a \le s \le b$$

*is an antiderivative for $f$; that is, $g'(x) = f(x)$. In other words,*

$$\frac{d}{dx}\int_a^x f(s)\,ds = f(x).$$

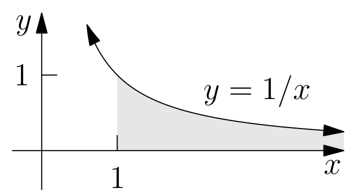
It is helpful to think of the function $g(x)$ as the "area function". That is, $g(x)$ is the area under the graph of $f$ between $a$ and $x$.

**Example 10.1.** Find the derivative of: (a) $g(x) = \int_1^x \ln(s)\,ds$ (b) $g(x) = \int_x^5 t^2\,dt$ (c) $g(x) = \int_0^{x^2} \sqrt{1+r^3}\,dr$ (d) $g(x) = \int_{e^x}^0 \sin^3 t\,dt$

## 11 Improper Integrals

Lec. #11

**Example 11.1.** Find the area of the infinite region under the graph of $y = 1/x^2$, to the right of $x = 1$.




We would like to calculate this area by evaluating the integral $\int_1^\infty \frac{1}{x^2}\,dx$. However, the definition of the definite integral only refers *finite* intervals. Integrals of this type are therefore said to be *improper*. The following give precise definitions that make sense of improper integrals.

**Definition 11.1.** $\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$, if the limit exists and $\int_a^t f(x)\,dx$ exists for all $t \ge a$.

**Definition 11.2.** $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$, if the limit exists and $\int_t^b f(x)\,dx$ exists for all $t \le b$.

Improper integrals are said to be *convergent* if the corresponding limit exists, and *divergent* if not.

**Definition 11.3.** $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$, if both $\int_{-\infty}^a f(x)\,dx$ and $\int_a^\infty f(x)\,dx$ are convergent.

We can apply the first definition to solve the previous example:

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left[1-\frac{1}{t}\right] = 1$$

so the area of this infinite region is (somewhat surprisingly) 1.

**Example 11.2.** For each of the following, determine whether the integral is convergent; if it is, evaluate the integral. (a) $\int_1^\infty \frac{1}{x^3}\,dx$ (b) $\int_1^\infty \frac{1}{x}\,dx$ (c) $\int_1^\infty e^{-x}\,dx$ (d) $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$ (e) $\int_{-\infty}^\infty x^3 e^{-x^4}\,dx$

**Example 11.3.** For what values of $p$ is the integral $\int_1^\infty x^{-p}\,dx$ convergent?

**Example 11.4.** The infinite region bounded by $y = 1/x$ and $y = 0$ to the right of $x = 1$ is revolved about the $x$-axis. Find the volume of revolution.

Lec. #12

### 11.1 Discontinuous integrands

Recall that $\int_a^b f(x)\,dx$ is only defined if $f$ is a continuous function. Therefore another kind of "improper" integral arises if $f$ is discontinuous. The following definition makes sense of this kind of integral.

**Definition 11.4.** If $f$ is continuous on $[a, b)$ and discontinuous at $b$ then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

provided the limit exists.

**Definition 11.5.** If $f$ is continuous on $(a, b]$ and discontinuous at $a$ then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

provided the limit exists.

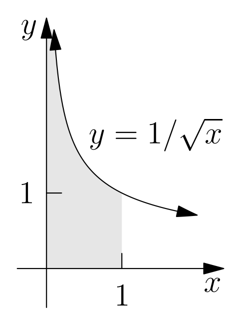
Again, if the corresponding limit exist we say the integral is *convergent* and otherwise *divergent*.

**Definition 11.6.** If $f$ is discontinuous at $c$ where $a < c < b$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

provided both integrals on the right are convergent.

**Example 11.5.** Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.




Since $f(x) = \frac{1}{\sqrt{x}}$ is discontinuous at $x = 0$, apply the second definition:

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0+} \int_t^1 x^{-1/2}\,dx = \lim_{t \to 0^+} \left[2x^{1/2}\right]_t^1 = \lim_{t \to 0^+} \left[2 - 2\sqrt{t}\right] = 2.$$

**Example 11.6.** Evaluate: (a) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{x}}\,dx$ (b) $\displaystyle\int_0^1 \ln x\,dx$ (c) $\displaystyle\int_0^1 \frac{1}{x}\,dx$

**Example 11.7.** The infinite region bounded by $y = 1/x$ and $y = 0$ betweem $x = 0$ and $x = 1$ is revolved about the $x$-axis. Find the volume of revolution.

**Example 11.8.** For what values of $p$ is the integral $\displaystyle\int_0^1 x^{-p}\,dx$ convergent?

## 12 Integrals of Rational Functions

Lec. #13

### 12.1 Partial Fractions

Recall that

$$\frac{1}{x-3} - \frac{1}{x+2} = \frac{(x+2)-(x-3)}{(x+2)(x-3)} = \frac{5}{x^2-x-6}.$$

If we needed to evaluate $\int \frac{6}{x^2 - x - 6}\,dx$ and we happened to know the above result, then we could do:

$$\int \frac{6}{x^2 - x - 6}\,dx = \int \left[\frac{1}{x-3} - \frac{1}{x+2}\right] dx = \ln|x-3| - \ln|x+2| + C.$$

This suggests a general strategy for integrating rational functions: factor the denominator and try writing the rational function as a some of simpler functions.

**Example 12.1.** Evaluate $\int \frac{1}{x^2 + 2x - 8}\,dx$.

The denominator factors as $(x+4)(x-2)$, and we suspect we can write

$$\frac{1}{x^2 + 2x - 8} = \frac{A}{x+4} + \frac{B}{x-2}$$

for some numbers $A$, $B$. To find these numbers, rewrite this with a common denominator:

$$\frac{1}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2} = \frac{A(x-2) + B(x+4)}{(x+4)(x-2)} = \frac{(A+B)x + (-2A+4B)}{(x+4)(x-2)}.$$

Equating numerators, we get

$$(A+B)x + (-2A+4B) = 1.$$

Since this equation must be true for all values of $x$, it must be true that

$$\begin{cases} A + B = 0 \\ -2A + 4B = 1. \end{cases}$$

This is a system of two linear equations in two unknowns, and can be solved to give $B = 1/6$, $A = -1/6$. Thus,

$$\int \frac{1}{x^2 + 2x - 8}\,dx = \int \left[\frac{-1/6}{x+4} + \frac{1/6}{x-2}\right] dx = -\tfrac{1}{6}\ln|x+4| + \tfrac{1}{6}\ln|x-2| + C.$$

**Example 12.2.** Evaluate: (a) $\int \frac{2x+3}{x^2 - 5x + 6}\,dx$ (b) $\int \frac{x}{x^2 + 7x + 10}\,dx$

## 12.2 Polynomial Long Division

Partial fractions will works for $\int \frac{P(x)}{Q(x)}\,dx$ where $P$ and $Q$ are polynomials with $\deg P < \deg Q$. We will only cover the case where $Q$ factors into a number of linear terms; you can find other cases in any good calculus text, if you ever need to.

If $\deg P \geq \deg Q$ it is necessary to first simply by applying polynomial long division.

**Example 12.3.** Evaluate $\int \frac{x^3 + 5}{x - 2}\,dx$.

First expand the rational function by applying polynomial long division:

$$\begin{array}{r|l}
 & x^2 + 2x + 4 \\
\hline
x - 2 & x^3 \qquad\qquad + 5 \\
 & -\, x^3 + 2x^2 \\
\hline
 & 2x^2 \\
 & -\, 2x^2 + 4x \\
\hline
 & 4x + 5 \\
 & -\, 4x + 8 \\
\hline
 & 13
\end{array}$$

So we get

$$\int \frac{x^3+5}{x-2}\,dx = \int \left[x^2+2x+4+\frac{13}{x-2}\right]\,dx = \tfrac{1}{3}x^3+x^2+4x+13\ln|x-2|+C.$$

**Example 12.4.** Evaluate: (a) $\displaystyle\int \frac{x^2+2x-8}{x+3}\,dx$ (b) $\displaystyle\int \frac{x^2}{x^2-9}\,dx$

Notice how long division reduces things to the case where deg $P <$ deg $Q$. Partial fractions is still sometimes necessary after this reduction.

## 12.3 Completing the Square

Lec. #14

With polynomial long division we can always eventually reduce a rational function to $\frac{P(x)}{Q(x)}$ where deg $P <$ deg $Q$. If $Q$ factors we can then apply partial fractions if necessary. The only case we haven't covered is where $Q$ does not factor.

**Example 12.5.** Evaluate $\displaystyle\int \frac{1}{x^2+4}\,dx$.

Here the denominator doesn't factor, but it looks a lot like something involving arctan. In fact,

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4(\frac{x^2}{4}+1)}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx.$$

Now make the substitution $u^2 = \frac{x^2}{4}$, so $u = \frac{x}{2}$ and $du = \frac{1}{2}\,dx$:

$$\frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{u^2+1}\cdot 2\,du = \frac{1}{2}\int \frac{1}{u^2+1}\,du = \tfrac{1}{2}\arctan u + C = \tfrac{1}{2}\arctan\left(\frac{x}{2}\right)+C.$$

The previous example generalizes to the following useful formula:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.$$

**Example 12.6.** Evaluate $\displaystyle\int \frac{1}{x^2+4x+6}\,dx$.

Again, the denominator doesn't factor (discriminant: $b^2-4ac=-8<0$). But complete the square:

$$\int \frac{1}{x^2+4x+6}\,dx = \int \frac{1}{(x+2)^2+2}\,dx$$

and make the substitution $u = x+2$, $du = dx$:

$$\int \frac{1}{(x+2)^2+2}\,dx = \int \frac{1}{u^2+2}\,du.$$

By the previous "useful formula", we get

$$\int \frac{1}{u^2+2}\,du = \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right)+C.$$

### 12.4 General Recipe

Where doing an integral like $\int \frac{P(x)}{Q(x)}\,dx$ the following general recipe is useful:

**Case 1: deg $P \geq$ deg $Q$.** Apply polynomial long division, then proceed to Case 2 if necessary.

**Case 2: deg $P <$ deg $Q$.** There are two subcases:

**Case (i): $Q$ factors.** Use partial fractions.

**Case (ii): $Q$ doesn't factor.** Complete the square and use substitutions.

The method of partial fractions itself has many subcases, depending on the factors of $Q$. We'll only deal with the case where all of the factors of $Q$ are simple, linear terms.

## 13 Integrals of Trig Functions

Lec. #15

Integrals involving trigonometric functions often need to be rewritten using trig identities so that an obvious substitution can be recognized.

**Example 13.1.** Evaluate $\displaystyle\int \cos^3 x\,dx$.

First peel off one power of $\cos x$:

$$\int \cos^3 x\,dx = \int \cos^2 x\ \cos x\,dx$$

and then write the rest in terms of $\sin x$, using the Pythagorean identity:

$$= \int (1-\sin^2 x)\ \cos x\,dx.$$

Now $u = \sin x$ is an obvious substitution:

$$= \int (1-u^2)\,du = u - \tfrac{1}{3}u^3 + C = \sin x - \tfrac{1}{3}\sin^3 x + C.$$

The same trick will work for any of the following

$$\int \sin^n x\,dx \qquad \int \sin^n x\cos^m x\,dx \qquad \int \cos^n x\,dx \qquad \int \cos^n x\sin^m x\,dx$$

provided $n$ is an odd number.

**Example 13.2.** Evaluate: (a) $\displaystyle\int \sin^5 x\,dx$ (b) $\displaystyle\int \sin^2 x\cos^3 x\,dx$

To integrate even powers of $\sin x$ or $\cos x$ you need to use double-angle identities:

**Example 13.3.** Evaluate $\displaystyle\int \sin^2 x\,dx$.

We can use a double-angle identity to rewrite this as

$$\int \sin^2 x\,dx = \int \tfrac{1}{2}(1-\cos 2x)\,dx.$$

This is now a straightforward antiderivative:

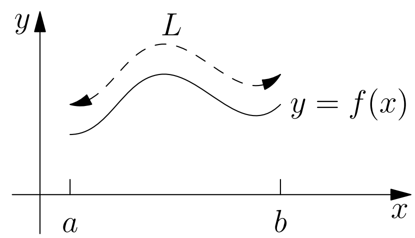
$$\tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) + C.$$

# 14 Applications

Lec. #16

## 14.1 Arc Length

**Problem:** Find the length of the curve given by the graph of $y = f(x)$ between $x = a$ and $x = b$.




**Answer:**

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

[derivation given in class]

**Example 14.1.** Find the arc length of the graph of $y = x^2$ between $x = 0$ and $x = 1$.

Applying the formula above gives

$$L = \int_0^1 \sqrt{1 + (2x)^2}\, dx.$$

With the substitution $u = 2x$ this becomes

$$L = \tfrac{1}{2} \int_0^2 \sqrt{1 + u^2}\, du.$$

The antiderivative can be found in a table of integrals (e.g. #21 in the table at the back of Stewart):

$$L = \frac{1}{2}\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_0^2 = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] \approx 1.48.$$

## 14.2 Average Value of a Function

**Problem:** Find the average value of $f(x)$ for $x$ between $a$ and $b$

**Answer:**

$$\bar{f} = f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

**Example 14.2.** Find the average value of $y = x^2$ for $x$ between 1 and 3.

Applying the formula above gives

$$\bar{f} = \frac{1}{3-1}\int_1^3 x^2\, dx = \frac{1}{2}\left(\frac{1}{3}x^3\right)\Big|_1^3 = \frac{13}{2}$$

**Example 14.3.** Find the average value of $\sin x$ for $x$ between: (a) 0 and $2\pi$ (b) 0 and $\pi$.

## 14.3 Center of Mass

**Problem:** Find the $x_{\text{cm}}$, the $x$-coordinate of the center of mass, for the plane region under the graph of $y = f(x)$ between $x = a$ and $x = b$. (The region would balance on a pin placed exactly at $x_{\text{cm}}$.)

**Answer:**

$$x_{\text{cm}} = \frac{\displaystyle\int_a^b x f(x)\, dx}{\displaystyle\int_a^b f(x)\, dx}$$

**Example 14.4.** Find the $x$-coordinate of the center of mass of the region under the graph of $y = x$ between $x = 0$ and $x = 1$ (i.e. a triangle).

$$x_{\text{cm}} = \frac{\displaystyle\int_0^1 x \cdot x\, dx}{\displaystyle\int_0^1 x\, dx} = \frac{\frac{1}{3}x^3\Big|_0^1}{\frac{1}{2}x^2\Big|_0^1} = \frac{1/3}{1/2} = \frac{2}{3}$$

## 14.4 Work

**Example 14.5.** Find the work required to compress a spring (with spring constant $k$) from its rest position ($x = 0$) to a final displacement of $X$.

When the spring is compressed an amount $x$, further compression by an infinitesimal amount $dx$ requires infinitesimal work

$$dW = F\, dx = (kx)\, dx.$$

Thus the total work to compress the spring from $x = 0$ to $x = X$ is

$$W = \int dW = \int_0^X kx\, dx = \tfrac{1}{2}kx^2\Big|_0^X = \tfrac{1}{2}kX^2$$

**Example 14.6.** A cylindrical tank of radius $r$ and height $h$ is to be filled with water. Find the total work required to fill the tank if the water source is level with the bottom of the tank.

When the tank is filled to height $y$, increasing the water level by an infinitesimal amount $dy$ requires the work needed to lift the volume of water

$$dV = \pi r^2\, dy$$

to a height $y$. If the density of water is $\rho$ then the mass of water to be lifted is

$$dm = \rho dV = \rho\pi r^2\, dy.$$

The infinitesimal amount of work required to lift this mass to height $y$ is

$$dW = gy\, dm = \rho g\pi r^2 y\, dy.$$

So the total work required to fill the tank to height $h$ is

$$W = \int dW = \int_0^h \rho g\pi r^2 y\, dy = \rho g\pi r^2 \left(\tfrac{1}{2}y^2\right)\Big|_0^h = \tfrac{1}{2}\rho g\pi r^2 h^2$$

**Example 14.7.** An inverted conical tank of height $h$ and radius $r$ is to be filled with water of density $\rho$. Find the total work required to fill the tank if the water source is level with the bottom of the tank.

## 14.5 Probability

A random variable $X$ is one whose value is the result of a random process, and therefore not known in advance. A random variable is frequently described in terms of its probability density function, $f(x)$, which has the interpretation that

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$

In order to be a well-defined probability density function, $f$ must satisfy two conditions:

$$1.\quad f(x) \geq 0 \qquad \text{and} \qquad 2.\quad \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

**Example 14.8.** Let

$$f(x) = \begin{cases} Ax(1-x) & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

For what value of $A$ is $f(x)$ a probability density function?

**Example 14.9.** Let $X$ be the length of time (in minutes) you are left on hold if you call Microsoft technical support; $X$ is a random variable with probability density

$$f(x) = \begin{cases} 10e^{-t/10} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

(a) Verify that $f$ is a probability density function.
(b) Find the probability that a randomly selected caller will be on hold longer than 15 minutes.

### 14.5.1 Average Values

In the previous example, one might wonder what is the *average* time that callers spend on hold. This question can be answered in terms of the probability density function.

In general, if $X$ is a random variable with probability density $f(x)$ then the average, or *mean* value of $X$ is given by

$$\bar{x} = \int_{-\infty}^{\infty} x f(x)\,dx.$$

**Example 14.10.** Find the average waiting time in the previous example.

More generally, the average value of a function $g(x)$ is given by

$$\bar{g} = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

For example the *variance*, $\sigma^2$ of $X$, is the average value of $(x-\bar{x})^2$:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx.$$

The *standard deviation*, $\sigma$, of $X$ is the square root of the variance.

**Example 14.11.** In the waiting time example above, find the standard deviation of the waiting time.

# 15 Differential Equations

## 15.1 Modeling with Differential Equations

Lec. #17

Suppose you have 1000 in the bank, earning 5% interest annually. In the first year the interest earned is $0.05 \times \$1000 = \$50$. The balance after one year is \$1050, so the interest earned in the second year is $0.05 \times \$1050 = \$52.50$, and so on.

As the balance grows, so does the interest earned per year (i.e., the *rate of growth* of the bank balance). In the first year the rate of growth is

$$0.05 \times 1000 = 50 \ (\$/\text{yr}).$$

In the second year the rate of growth is

$$0.05 \times 1050 = 52.5 \ (\$/\text{yr}).$$

We can model this growth process mathematically as follows. Let the function $x(t)$ be the bank balance at time $t$ (so $x(0) = 1000$). At any given time, the rate of growth of $x$ is $0.05x$ (in \$/yr). We can express this fact by the equation

$$\frac{dx}{dt} = 0.05x \qquad \text{or} \qquad x'(t) = 0.05x(t).$$

This is called a *differential equation*: an equation relating a function $x(t)$ to its derivative $x'(t)$ (and potentially also to its higher derivatives $x''(t)$, etc). The solution of such an equation consists of the unknown function $x(t)$ that satisfies it.

If we could find a function $x(t)$ such that $x' = 0.05x$ (i.e. satisfies the differential equation) then this function could be used to predict the bank balance at any time in the future.

In fact, we can easily recognize such a function: $x(t) = Ae^{0.05t}$ (where $A$ is an arbitrary constant). The value of $A$ must be chosen to match the "initial condition" $x(0) = 1000$:

$$x(0) = Ae^0 = A = 1000.$$

So the solution is

$$x(t) = 1000e^{0.05t}.$$

For example, the bank balance in 10 years will be $x(10) = 1000e^{0.05(10)} \approx \$1648.72$.

Another process that can be modelled by a similar differential equation is radioactive decay. Suppose you have 1 kg of a radioactive substance. Through radioactive decay, a certain fraction (say, 0.0002) of the mass disappears every year. As the mass gradually decays, the amount that decays each year becomes less, whereas the proportion of the current mass that decays each year, 0.0002, is constant.

Let $x(t)$ be the mass remaining after time $t$. Then at any given time the rate of change of $x$ is $-0.0002x$. This can be expressed as the differential equation

$$x' = -0.0002x.$$

Its solution is

$$x(t) = Ae^{-0.0002t}$$

where $A = 1$ to satisfy the initial condition $x(0) = 1$.

## 15.2 Separable Equations

Differential equations can be very difficult to solve, but we only need to consider the relatively easy case of *separable* equations.

**Example 15.1.** Solve the differential equation $\frac{dx}{dt} = 0.05x$ (from the compound interest problem in the previous section).

Abusing notation somewhat, multiply both sides by $dt$ to obtain

$$dx = 0.05xdt.$$

Then divide both sides by $x$:

$$\frac{dx}{x} = 0.05dt.$$

Now integrate both sides:

$$\int \frac{dx}{x} = \int 0.05dt \implies \ln x = 0.05t + C$$

where $C$ is an arbitrary "constant of integration" that must be included as it will play an important role. We want to solve for $x(t)$ so exponentiate both sides:

$$x = e^{0.05t+C} = e^{0.05t}e^{C}.$$

Notice that $e^C$ is just a constant, but it is kind of awkward so it is convenient to define a new constant $A = e^C$. Then

$$x(t) = Ae^{0.05t}$$

is the most general solution of the differential equation.

Notice that because the constant $A$ is arbitrary, rather than a single solution we have found a whole *family* of solutions — one for each possible value of $A$. In applications, $A$ usually takes a specific value chosen so that $x(t)$ satisfies an "initial condition" that specifies a value of $x(0)$.

A differential equation for a function $x(t)$ is called *separable* if all quantities involving $x$ can be brought to one side of the equation, with all quantities involving $t$ on the other side. This step is critical because it permits us to then integrate both sides of the equation.

**Example 15.2.** Solve the differential equation $\frac{dy}{dx} = xy$.

This equation is separable: it can be written as

$$\frac{dy}{y} = x\,dx.$$

Integrating both sides we get

$$\int \frac{dy}{y} = \int x\,dx \implies \ln y = \tfrac{1}{2}x^2 + C \implies y = e^{\frac{1}{2}x^2+C} = e^{\frac{1}{2}x^2}e^{C} = Ae^{\frac{1}{2}x^2}$$

where $A = e^C$ is an arbitary constant.

**Example 15.3.** Solve the differential equation: (a) $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$ (b) $\frac{dy}{dx} = y^2 \sin x$

## 15.3 Mixing Problems

Lec. #18

**Example 15.4.** A 1000 L tank is initially full of pure water. Water containing 5 mg/L chlorine is added to the tank at a rate of 80 L/min. The tank is kept well-mixed, and the mixed water flows out a drain at a rate of 80 L/min (so that the water volume in the tank is kept constant). Find the chlorine concentration in the tank as a function of time.

Let $y(t)$ be the amount [mg] of chlorine in the tank at time $t$. Then

$$\frac{dy}{dt} = \text{“rate in”} - \text{“rate out”}$$

where

$$\text{“rate in”} = 80\,\text{L/min} \times 5\,\text{mg/L} = 400\,\text{mg/min}$$

and similarly

$$\text{“rate out”} = 80 \times \frac{y}{1000} = 0.08y.$$

Therefore $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = 400 - 0.08y.$$

This equation is separable, so we can solve it easily:

$$\begin{aligned} \int \frac{dy}{400 - 0.08y} = \int dt &\implies \frac{1}{-0.08}\ln(400 - 0.08y) = t + C \\ &\implies 400 - 0.08y = e^{-0.08(t+C)} = Ae^{-0.08t} \\ &\implies y = \frac{1}{0.08}\left(400 - Ae^{-0.08t}\right) = 5000 - Be^{-0.08t} \end{aligned}$$

The initial condition requires that $y(0) = 0$ so

$$5000 - Be^{0} = 0 \implies B = 5000$$

so the final solution is

$$y(t) = 5000 - 5000e^{-0.08t} \quad [\text{mg}].$$

The concentration is given by

$$\frac{y(t)}{1000} = 5(1 - e^{-0.08t}) \quad [\text{mg/L}].$$

Notice that as $t \to \infty$ the concentration approaches 5 mg/L, as might be expected. The time required to approach this equilibrium concentration is related to the constant 0.08. For example, the time it takes to reach a concentration of 4 mg/L is given by

$$4 = 5(1 - e^{-0.08t}) \implies e^{-0.08t} = 0.2 \implies -0.08t = \ln(0.2) \implies t = \frac{\ln(0.2)}{-0.08} \approx 20\,\text{min}.$$

**Example 15.5.** A 400 L tank is initially filled with water containing salt at a concentration of 2 g/L. Fresh water is slowly added to the tank at a rate of 8 L/hr. The tank is kept well-mixed, and the mixed water flows out a drain at the rate of 8 L/hr. How long does it take for the salt concentration on the tank to fall to half its initial value?

Let $y(t)$ be the amount [g] of salt in the tank at time $t$. Then $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = -8 \times \frac{y}{400} = -0.02y.$$

This equation is separable:

$$\int \frac{dy}{y} = \int -0.02\,dt \implies \ln y = -0.02t + C$$
$$\implies y = e^{-0.02t+C} = e^{-0.02t}e^{C} = Ae^{-0.02t}$$

Imposing the "initial condition" requires

$$2 \times 400 = Ae^0 \implies A = 800$$

so that

$$y(t) = 800e^{-0.02t} \quad \text{[g]}.$$

The time required for $y$ to fall to half its initial value (i.e. 400 g) is given by

$$400 = 800e^{-0.02t} \implies -0.02t = \ln(0.5) \implies t = \frac{\ln(0.5)}{-0.02} \approx 34.7\,\text{h}.$$

Lec. #19

## 15.4 Exponential growth and decay

The previous example illustrates a common model that arises in many different contexts.

For example, in radioactive decay, a given mass $m$ of a radioactive substance decreases in time according to

$$\boxed{\frac{dm}{dt} = -km}$$

That is, a certain fraction $k$ of the mass decays per unit time. As the mass decreases, the rate of decay decreases in proportion. This differential equation is separable, and its solution is

$$\boxed{m(t) = m_0 e^{-kt}}$$

where $m_0$ is the mass at time 0.

The decay constant $k$ is related to the half-life, $t_{1/2}$, of the substance. In fact the half-life is given by the condition

$$\frac{1}{2}m_0 = m_0 e^{-kt} \implies \ln(1/2) = -kt \implies t = \frac{\ln(1/2)}{-k} = \frac{\ln(2)}{k}$$

so

$$\boxed{t_{1/2} = \frac{\ln 2}{k}}$$

**Example 15.6.** 800 g of a radioactive substance is left on a shelf for 3 years. At the end of this time there is 750 g of the substance remaining. What is the half-life of this substance.

I similar differential equation models exponential growth. For example, if in a given bacterial culture the reproduction rate is $k$ [individuals per unit time, per individual in the culture] then the population satisfies

$$\frac{dx}{dt} = kx$$

whose solution is

$$x(t) = x_0 e^{kx}$$

where $x_0$ is the population at time 0.

**Example 15.7.** In 1980 the world population was 4.2 billion; in 2005 it was 6.4 billion. Assuming an exponential growth model, in what year will the population be 10 billion?

Lec. #20

# 16 Infinite Series

## 16.1 Sequences

A *sequence* is an infinitely long list of numbers. For example,

$$\{a_n\} = \{1, 2, 3, 4, \ldots\}$$

is a sequence. The notation $\{a_n\}$ refers to a sequence whose $n$'th term is $a_n$. Sometimes we have an explicit formula for $a_n$, e.g.

$$\left\{\frac{1}{n}\right\} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}$$

and

$$\left\{\frac{n}{1+n}\right\} = \{\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \ldots\}$$

If $\lim_{n\to\infty} a_n$ exists then the sequence $\{a_n\}$ is said to be *convergent*. The previous two examples are both convergent sequences, since for the first,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

and for the second,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{1+n} = 1.$$

Clearly the sequence $\{1, 2, 3, 4, \ldots\}$ is not convergent.

A sequence $\{a_n\}$ can also be defined recursively, as in the famous Fibonacci sequence:

**Example 16.1.** Let $x_1 = x_2 = 1$ and $x_n = x_{n-2} + x_{n-1}$ for all $n \geq 3$. Each subsequent term in the sequence is defined in terms of the previous terms:

$$\begin{aligned}
x_1 &= 1 \\
x_2 &= 1 \\
x_3 &= x_1 + x_2 = 1 + 1 = 2 \\
x_4 &= x_2 + x_3 = 1 + 2 = 3 \\
x_5 &= x_3 + x_4 = 2 + 3 = 5 \ldots
\end{aligned}$$

So the Fibonacci sequence is $\{x_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$. (Pretty clearly, this sequence does not converge.)

Lec. #21

## 16.2 Series

Given a sequence $\{a_n\}$, if we add all the terms in the sequence we get an *infinite series*:

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

For example, the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

arises in many applications. But how is it possible to add up (or even make sense of) the sum of infinitely many terms? As usual in calculus, We can approach this problem using limits. The idea is to first define $s_n$ to be the sum of the first $n$ terms of the series:

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

As $n$ increases, more and more terms are included in this sum. We then define the sum of the infinite series to be

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

If this limit exists then we say the series is *convergent*; otherwise, *divergent*.

**Example 16.2.** To calculate $\sum_{n=1}^{\infty} \frac{1}{2^n}$ we first calculate the partial sums:

$$s_n = \sum_{i=1}^{n} \frac{1}{2^i} = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i.$$

There is a useful formula for a sum of the form $\sum_{i=1}^{n} r^i$, which be derived quickly as follows. Let

$$S = \sum_{i=1}^{n} r^i = r + r^2 + r^3 + \cdots r^n.$$

Then

$$rS = r^2 + r^2 + \cdots + r^n + r^{n+1}.$$

If we subtract this two equations most of the terms cancel, and we're left with:

$$S - rS = r - r^{n+1} \implies S = \frac{r - r^{n+1}}{1-r}.$$

So we have

$$\boxed{\sum_{i=1}^{n} r^i = \frac{r - r^{n+1}}{1-r}}$$

and a similar derivation can be used to show that

$$\boxed{\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1-r}}$$

Going back to our problem, we can now calculate the $n$'th partial sum:

$$s_n = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{(1/2) - (1/2)^{n+1}}{1-(1/2)}.$$

Letting $n \to \infty$ we find that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{(1/2) - (1/2)^{n+1}}{1-(1/2)} = \frac{(1/2)}{1-(1/2)} = 1.$$

## 16.3 Geometric Series

The series in the previous example is called a *geometric series*, and our result can be generalized as follows:

**Theorem 16.1.** *If $|r| < 1$ then the geometric series*

$$\sum_{n=1}^{\infty} r^n = r + r^2 + r^3 + \cdots = \frac{r}{1-r}$$

*and*

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$$

*are convergent. If $|r| \geq 1$ then these series are divergent.*

**Example 16.3.** Decide whether each of the following series is convergent, and if possible evaluate the infinite sum. (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{\pi^n}$ (c) $1 + (-1) + 1 + (-1) + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n$

## 16.4 Tests for Convergence

We will need ways to decide if a given infinite series $\sum_{n=1}^{\infty} a_n$ is convergent, without calculating the partial sums, which can be very difficult except in special cases.

**Theorem 16.2.** *If $\lim_{n\to\infty} a_n \neq 0$ or does not exist then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.*

**Example 16.4.** The following series are divergent: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$ (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ (c) $\displaystyle\sum_{n=1}^{\infty} \cos(n)$

Note: the previous theorem tells us nothing in the event that $\lim_{n\to\infty} a_n = 0$. That is, this alone does *not* guarantee the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, we will prove shortly that the *harmonic series*

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \quad \text{is divergent}}$$

even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

**Theorem 16.3** (Integral Test)**.** *If $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ then the series $\sum_{n=1}^{\infty} a_n$ with $a_n = f(n)$ is:*

- *convergent if $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, and*
- *divergent if $\displaystyle\int_1^{\infty} f(x)\,dx$ is diveregent.*

For example, since $f(x) = \frac{1}{x}$ is continuous, positive and decreasing on $[1, \infty)$ we now know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent only if $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent. We have seen that this improper integral diverges, so the harmonic series is divergent.

**Example 16.5.** Which of the following series converge: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

**Example 16.6.** For what values of $p$ does the $p$-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

**Theorem 16.4** (Ratio Test)**.** *Given a series* $\sum_{n=1}^{\infty} a_n$*, define* $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. *Then the series is*

- *convergent if* $L < 1$*,*
- *divergent if* $L > 1$*, and*
- *could be either convergent or divergent if* $L = 1$*.*

**Example 16.7.** Use the ratio test to investigate the converge of the series: (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(1+n)^2}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

## 16.5 Power Series

If $x$ is a real number between 0 and 1 then the geometric series $\sum_{n=1}^{\infty} x^n$ converges. Therefore we can consider this series as a function of $x$. In fact we know that the sum of the series is

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}.$$

We can turn this around: given the function $f(x) = 1/(1-x)$, we can represent $f$ in terms of a series. This turns out to be a very useful way to represent functions.

For a given sequence $\{c_n\}$ the function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

is called a *power series* with coefficients $c_n$.

More generally, the function

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is the power series *centered at* $a$, with coefficients $c_n$.

The central question about a given power series is the values of $x$ for which the series converges (since this determines the domain of the series, considered as a function of $x$). The ratio test is usually the best way to determine the interval of convergence for a power series.

**Example 16.8.** Find the interval of convergence for the power series: (a) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$ (b) $f(x) = \displaystyle\sum_{n=1}^{\infty} \sqrt{n}x^n$ (c) $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n}}$

For a power series centered at $a$, the interval of convergence is always of the form $|x-a|<R$, where $R$ is the *radius of convergence.* Find the radius of convergence for each of the series in the previous example.

It can be very useful to represent a function in terms of a power series. This is basically because a power series is essentially a polynomial, and polynomials are easy to manipulate.

One important power series representation we've seen already is

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$$

which is just a geometry series. This can be used to find power series for other functions, for example:

$$\frac{1}{1-x^2}=\sum_{n=0}^{\infty}(x^2)^n=\sum_{n=0}^{\infty}x^{2n}$$

$$\frac{1}{1+x}=\frac{1}{1+(-x)}\sum_{n=0}^{\infty}(-x)^n=\sum_{n=0}^{\infty}(-1)^nx^n$$

$$\frac{x}{1-x}=x\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}xx^n=\sum_{n=0}^{\infty}x^{n+1}=\sum_{n=1}^{\infty}x^n$$

**Theorem 16.5.** *Suppose the power series*

$$\sum_{n=0}^{\infty}c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+\cdots$$

*is differentiable on the interval* $(a-R,a+R)$*. Then*

- $f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\cdots=\sum_{n=1}^{\infty}nc_n(x-a)^{n-1}$
- $\int f(x)\,dx=c_0(x-a)+\frac{c_1}{2}(x-a)^2+\frac{c_2}{3}(x-a)^3+\cdots+C=\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}+C$

*and both these series have radius of convergence* $R$*.*

In other words, power series can be differentiated (or integrated) term by term, as if they were just polynomials.

**Example 16.9.**

## 16.6 Taylor and Maclaurin Series

Recall the linearization of a function $f(x)$ near $x=a$:

$$P_1(x)=L(x)=f(a)+f'(a)(x-a)$$

which is just the equation of the tangent line to the graph of $f$ at $x=a$. $L(x)$ can be defined as the (unique) linear function that has the same value and derivative at $a$ as $f(x)$.

The linear approximation can be improved by adding a quadratic correction term. The *quadratic approximation* of $f$ at $a$ is the (unique) quadratic function that has the same value, derivative and second derivative at $a$ as $f(x)$.

$$P_2(x)=f(a)+f'(a)(x-a)+\tfrac{1}{2}f''(a)(x-a)^2$$

**Example 16.10.** Use a quadratic approximation to estimate $\sqrt{4.5}$.

This can be generalized to define the $n$-degree *Taylor polynomial* for $f(x)$ based at $x = a$:

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \cdots + \tfrac{1}{n!}f^{(n)}(a)(x-a)^n \end{aligned}$$

The function $P_n(x)$ is an $n$-th degree polynomial that approximates $f(x)$ near $x = a$. The greater the value of $n$, the better the approximation.

By including an infinite number of terms in the Taylor polynomial, we arrive at a *Taylor series*:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k} = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2 + \cdots$$

which is an *exact* representation (i.e. no longer an approximation) of $f(x)$, provided this infinite series converges. In the case $a = 0$ the Taylor series is sometimes called a *Maclaurin series*:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k}$$

**Example 16.11.** Find Maclaurin series for the functions: (a) $e^x$ (b) $\sin x$ (c) $xe^{-x}$

Taylor and Maclaurin series are very useful, both for representing functions that are difficult or impossible to represent in another way, and to obtain approximate solutions to problems (by truncating the series to a finite number of terms).

**Example 16.12.** Find a Maclaurin series for $e^{-x^2}$.

We have the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

so that

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^{2n} \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \end{aligned}$$

**Example 16.13.** Find a Maclaurin series for $\int e^{-x^2}\,dx$.

Integrating the Maclaurin series in the previous example, term by term, we get

$$\begin{aligned} \int e^{-x^2}\,dx &= x - \frac{1}{3}x^3 + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}x^{2n+1} + C \end{aligned}$$

**Example 16.14.** Use a Maclaurin series to evaluate $\int_0^{0.5} e^{-x^2}\, dx$ to 3 decimal places.

From the previous example we have

$$\int_0^{0.5} e^{-x^2}\, dx = \left[x - \frac{1}{3}x^3 + \frac{1}{5}\frac{x^5}{2!} - \frac{1}{7}\frac{x^7}{3!} + \cdots\right]_0^{0.5}$$
$$= (0.5) - \frac{1}{3}(0.5)^3 + \frac{1}{5}\frac{(0.5)^5}{2!} - \frac{1}{7}\frac{(0.5)^7}{3!} + \cdots$$

Truncating the series to different numbers of terms gives the following:

| # of terms | approx. sum of series |
|---|---|
| 2 | 0.4583 |
| 3 | 0.4615 |
| 4 | 0.4613 |
| 5 | 0.4613 |

So, accurate to 3 decimal places, the value of the definite integral is 0.4613.

**Example 16.15.** Use a Maclaurin series to evaluate $\displaystyle\int_0^{0.2} \frac{\sin x}{x}\, dx$ to 3 decimal places.

# 17 Approximate Integration (Quadrature)

We have seen many examples where the solution of a scientific problem can be represented in terms of a definite integral. If that integral can be evaluated, then the problem is solved. However, some integrals can be difficult or impossible to evaluate exactly. In such cases, it is often sufficient to obtain a very good approximation (say, to 6 decimal places) of the integral. For this, there are many methods available. In the previous section we saw how Taylor and Maclaurin series can be used to approximate integrals. In this section we investigate some of the other methods available.

## 17.1 Midpoint Rule

Recall the definition of the definite integral as the limiting sum of the areas of rectangles:

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\sum_{i=1}^n f(x_i)\Delta x\right].$$

An obvious way to approximate the integral is simply to evaluate $S_n$ for some large $n$ (rather than take a limit). A slight improvement on this is to evaluate the height of the $i$'th rectangle at the midpoint, $\bar{x}_i$, (rather an the endpoint) of the $i$'th interval:

$$\int_a^b f(x)\, dx \approx \sum_{i=1}^n f(\bar{x}_i)\,\Delta x = \Delta x[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)]$$

where $\Delta x = (b-a)/n$ and

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}.$$

**Example 17.1.** Use the midpoint rule with $n = 4$ to approximate $\displaystyle\int_0^1 x^2\, dx$.

## 17.2 Trapezoidal Rule

Rather than subdivide the area under $y = f(x)$ into rectangles, we can clearly do a better job by subdividing into trapezoids. This results in the formula

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

**Example 17.2.** Use the trapezoidal rule with $n = 4$ to approximate $\int_0^1 x^2\,dx$.

## 17.3 Simpson's Rule

To more accurately approximate the shape of $y = f(x)$ we can subdivide the curve into parabolic (rather than straight-line) segments. This results in

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x)_3 + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Note that Simpson's rule is only defined if $n$ is even.

**Example 17.3.** Use the Simpson's rule with $n = 4$ to approximate $\int_0^1 x^2\,dx$.

## 17.4 Other methods

Integration plays such a central role in science (physics, chemistry and engineering especially) that accurate and efficient methods of approximate integration are constantly being developed. The methods dicussed above are only the most basic; much more sophisticated methods are available.

The TI-series of calculators use Gaussian quadrature, which is highly accurate...

Integrals arising in physics and economics are often evaluated used Monte-Carlo integration, which is most useful in higher dimensions (Calculus IV)...