THOMPSON RIVERS UNIVERSITY

MATH 124
Calculus II

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MIDTERM EXAM #2
SOLUTIONS

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Instructions:

1. Read all instructions carefully.
2. Read the whole exam before beginning.
3. Make sure you have all 6 pages.
4. Organization and neatness count.
5. You must clearly show your work to receive full credit.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		9
2		4
3		5
4		9
5		6
TOTAL:		33

/9

Problem 1: Evaluate the following integrals.

(a) $\displaystyle\int \frac{x+4}{x^2+5x-6}\,dx$

SOLUTION:

$$\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} = \frac{A(x-1)+B(x+6)}{(x+6)(x-1)}$$

$$\begin{cases} A+B=1 \\ -A+6B=4 \end{cases} \implies A=\tfrac{2}{7},\ B=\tfrac{5}{7}$$

$$\frac{2}{7}\int\frac{dx}{x+6} + \frac{5}{7}\int\frac{dx}{x-1} = \boxed{\frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C}$$

(b) $\displaystyle\int \frac{x^2}{x^2+9}\,dx$

SOLUTION:

$$\int\frac{x^2+9-9}{x^2+9}\,dx = \int\left(1-\frac{9}{x^2+9}\right)dx = \int 1\,dx - 9\int\frac{dx}{x^2+3^2}$$

$$= x - 9\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C = \boxed{x - 3\tan^{-1}(x/3) + C}$$

(c) $\displaystyle\int \cos^5(2x)\,dx$

SOLUTION:

$$\int\cos^4(2x)\cos(2x)\,dx = \int\left[1-\sin^2(2x)\right]^2\cos(2x)\,dx$$

substitution: $u=\sin(2x) \implies du = 2\cos(2x)\,dx$

$$\int\left[1-u^2\right]^2\tfrac{1}{2}du = \tfrac{1}{2}\int(1-2u^2+u^4)\,du = \tfrac{1}{2}\left(u-\tfrac{2}{3}u^3+\tfrac{1}{5}u^5\right)+C$$

$$= \boxed{\tfrac{1}{2}\sin(2x) - \tfrac{1}{3}\sin^3(2x) + \tfrac{1}{10}\sin^5(2x) + C}$$

/4

Problem 2: (a) Formulate (but *do not evaluate*) a definite integral whose value gives the length of the curve given by the graph of $y = \ln x$ for $1 \le x \le 2$.

SOLUTION:

$$y' = \frac{1}{x} \implies L = \int_1^2 \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$$

(b) Which of the formulas in the table of integrals (on the back page) could be used to evaluate the integral in part (a)? [Give just the number of the formula.]

SOLUTION:

$$L = \int_1^2 \sqrt{\frac{x^2+1}{x^2}}\, dx = \int_1^2 \frac{\sqrt{x^2+1}}{x}\, dx \implies \text{formula \#7 (with } a = 1)$$

/5

Problem 3: Solve the differential equation

$$\frac{dy}{dx} = \frac{2xy}{x^2+1}$$

with the initial condition $y(0) = 1$.

SOLUTION:

$$\int \frac{dy}{y} = \int \frac{2x}{x^2+1}\, dx \implies \ln y = \ln(x^2+1) + C$$

$$\implies y = e^{\ln(x^2+1)} e^C = A(x^2+1)$$

$$y(0) = 1 = A(0^2+1) \implies A = 1 \implies \boxed{y(x) = x^2 + 1}$$

/9

Problem 4: For each of the following series, determine whether the series is convergent; if it is, evaluate the infinite sum.

(a) $1 - 1 + 1 - 1 + \cdots$

SOLUTION:

$$a_n = (-1)^n$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \text{ does not exist (hence } \neq 0)$$

$$\text{Therefore the series is } \boxed{\text{divergent.}}$$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{3}{2^n}$

SOLUTION:

$$\sum_{n=0}^{\infty} (-1)^n \frac{3}{2^n} = 3 \sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n = 3 \cdot \frac{1}{1 - (-1/2)} = \boxed{2}$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 n!}{(n+2)!}$

SOLUTION:

$$a_n = \frac{n^2 n!}{(n+2)!} = \frac{n^2 n!}{(n+2)(n+1)n!} = \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = 1 \neq 0$$

$$\text{Therefore the series is } \boxed{\text{divergent.}}$$

/6

Problem 5: A 1000 L tank holds water that initially contains no chlorine. Water containing 3 mg/L chlorine is added to the tank at the rate of 100 L/min. The tank is well mixed, and the mixed solution is drained from the tank at the rate of 100 L/min.

(a) Let $x(t)$ be the amount (in mg) of chlorine in the tank at time t. Show that $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = 300 - 0.1x.$$

SOLUTION:

$$\begin{aligned}\frac{dx}{dt} &= \text{“rate in”} - \text{“rate out”} \\ &= (3\,\text{mg/L} \times 100\,\text{L/min}) - \left(\frac{x}{1000}\right) \cdot 100 \\ &= 300 - 0.1x\end{aligned}$$

(b) Solve the differential equation to find $x(t)$.

SOLUTION:

$$\begin{aligned}\int \frac{dx}{300 - 0.1x} = \int dt &\implies \frac{1}{-0.1}\ln(300 - 0.1x) = t + C \\ &\implies \ln(300 - 0.1x) = -0.1t + B \\ &\implies 300 - 0.1x = e^{-0.1t}e^{B} = Ae^{-0.1t} \\ &\implies -0.1x = Ae^{-0.1t} - 300 \\ &\implies x(t) = Ke^{-0.1t} + 3000\end{aligned}$$

$$x(0) = 0 = Ke^{0} + 3000 \implies K = -3000$$

$$\implies \boxed{x(t) = 3000 - 3000e^{-0.1t}}$$

Trigonometric Identities:

$$\sin^2 x + \cos^2 x = 1$$
$$\sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos(2x)$$
$$\cos^2 x = \tfrac{1}{2} + \tfrac{1}{2}\cos(2x)$$

Table of Integrals:

1. $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
2. $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$
3. $$\int \frac{dx}{x\sqrt{x^2 + a^2}} = -\frac{1}{a}\ln\left|\frac{\sqrt{x^2 + a^2} + a}{x}\right| + C$$
4. $$\int \frac{dx}{x^2\sqrt{x^2 + a^2}} = -\frac{x^2 + a^2}{a^2 x} + C$$
5. $$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C$$
6. $$\int \frac{\sqrt{x^2 + a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 + a^2}}{x} + \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$
7. $$\int \frac{\sqrt{x^2 + a^2}}{x}\, dx = \sqrt{x^2 + a^2} - a\ln\left|\frac{a + \sqrt{x^2 + a^2}}{x}\right| + C$$