THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus 2

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MIDTERM EXAM #2
SOLUTIONS

28 March 2019 08:30–09:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		5
3		5
4		5
5		5
TOTAL:		32

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Problem 1: Evaluate the following:

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(a) $\displaystyle\int \frac{x^4}{x^2-1}\,dx$

Polynomial long division gives $\dfrac{x^4}{x^2-1} = x^2+1+\dfrac{1}{x^2-1}$. Expand in partial fractions:

$$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$\begin{array}{l} x=-1 \implies 1 = A(-1-1)+0 \implies A=-1/2 \\ x=1 \implies 1 = 0+B(1+1) \implies B=1/2 \end{array} \implies \frac{1}{x^2-1} = \frac{-1/2}{x+1} + \frac{1/2}{x-1}$$

So finally:

$$\begin{aligned} \int \frac{x^4}{x^2-1}\,dx &= \int \left(x^2+1-\frac{1/2}{x+1}+\frac{1/2}{x-1}\right)dx \\ &= \boxed{\tfrac{1}{3}x^3 + x - \tfrac{1}{2}\ln|x+1| + \tfrac{1}{2}\ln|x-1| + C} \end{aligned}$$

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(b) $\displaystyle\int x^2 \sin x\,dx$

Integrate by parts: $\begin{array}{ll} u = x^2 & dv = \sin x\,dx \\ du = 2x\,dx & v = -\cos x \end{array}$

$$\implies \int x^2\sin x\,dx = -x^2\cos x - \int -2x\cos x\,dx = -x^2\cos x + \int 2x\cos x\,dx$$

By parts again: $\begin{array}{ll} u = 2x & dv = \cos x\,dx \\ du = 2\,dx & v = \sin x \end{array}$

$$\begin{aligned} \implies -x^2\cos x + \int 2x\cos x\,dx &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx \\ &= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + C} \end{aligned}$$

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(c) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} \cos^3 x\,dx &= \int_{-\pi/2}^{\pi/2} \cos^2 x \cos x\,dx \\ &= \int_{-\pi/2}^{\pi/2} (1-\sin^2 x)\cos x\,dx \qquad (u=\sin x,\ du = \cos x\,dx) \\ &= \int_{-1}^{1} (1-u^2)du \\ &= u - \tfrac{1}{3}u^3\Big|_{-1}^{1} = \boxed{\frac{4}{3}} \end{aligned}$$

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Problem 2: Consider the graph of $y = \ln x$ for $1 \le x \le 3$ (see the figure below).

(a) Write (but do not evaluate) a definite integral that represents the length of this curve.

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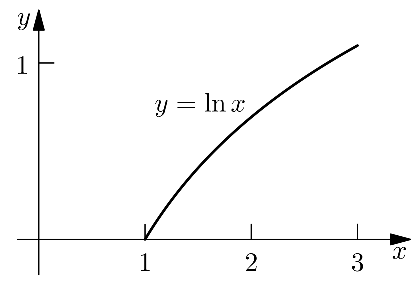

With $f(x) = \ln x$:

$$L = \int_1^3 \sqrt{1 + f'(x)^2}\, dx = \int_1^3 \sqrt{1 + \left(\frac{1}{x}\right)^2}\, dx$$
$$= \boxed{\int_1^3 \sqrt{1 + \frac{1}{x^2}}\, dx}$$

(b) Use Simpson's rule (with $n = 4$ intervals) to approximate the value of the integral in part (a).

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Let $g(x) = \sqrt{1 + \dfrac{1}{x^2}}$ and $\Delta x = \dfrac{3-1}{4} = \dfrac{1}{2}$. Then by Simpson's rule:

$$\int_1^3 g(x)\, dx \approx \frac{\Delta x}{3}\big[g(1) + 4g(1.5) + 2g(2) + 4g(2.5) + g(3)\big]$$
$$= \frac{1}{6}\left[\sqrt{1 + \frac{1}{1^2}} + 4\sqrt{1 + \frac{1}{1.5^2}} + 2\sqrt{1 + \frac{1}{2^2}} + 4\sqrt{1 + \frac{1}{2.5^2}} + \sqrt{1 + \frac{1}{3^2}}\right] \approx \boxed{2.303}$$

Wolfram Alpha shows the exact value is pretty close to our result:

$$\int_1^3 \sqrt{1 + \frac{1}{x^2}}\, dx = \sqrt{10} - \sqrt{2} + \ln 3 + \ln\frac{1 + \sqrt{2}}{1 + \sqrt{10}} \approx 2.3020$$

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Problem 3: Find the function $y(x)$ that satisfies the following initial value problem:

$$y' = \frac{y \ln y}{1 + x^2}, \quad y(0) = e^2$$

Separate variables and integrate:

$$\int \frac{dy}{y \ln y} = \int \frac{dx}{1 + x^2}$$

Substitute $u = \ln y$, $du = \dfrac{1}{y}\, dy$:

$$\int \frac{du}{u} = \int \frac{dx}{1 + x^2} \implies \ln|u| = \ln|\ln y| = \operatorname{atan} x + C$$

Solve for y:

$$|\ln y| = e^{\operatorname{atan} x} \underbrace{e^C}_{A} = Ae^{\operatorname{atan} x} \implies y = \exp\left(Ae^{\operatorname{atan} x}\right) = e^{Ae^{\operatorname{atan} x}}$$

Impose the initial value to determine A:

$$y(0) = e^2 = e^{Ae^{\operatorname{atan} 0}} = e^{Ae^0} = e^A \implies A = 2$$

so finally:

$$\boxed{y(x) = e^{2e^{\operatorname{atan} x}}}$$

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Problem 4: For each of the following improper integrals, either evaluate the integral or show that the integral diverges.

(a) $\displaystyle\int_0^\infty xe^{-x}\,dx$

Integrate by parts:

$$\begin{array}{ll} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{array} \implies \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$$

Now deal with the improper integral via a limit:

$$\begin{aligned} \int_0^\infty xe^{-x}\,dx &= \lim_{b\to\infty} \int_0^b xe^{-x}\,dx \\ &= \lim_{b\to\infty} \Big[-xe^{-x} - e^{-x} \Big]_0^b \\ &= 1 - \underbrace{\lim_{b\to\infty} \Big(be^{-b} + e^{-b} \Big)}_{0 \text{ (by l'Hopital)}} = \boxed{1} \end{aligned}$$

(b) $\displaystyle\int_{-1}^1 \frac{dx}{x^{1/3}}\,dx$

The function $\dfrac{1}{x^{1/3}}$ blows up at $x = 0$ so we need to split up the integral and deal with the blowup via limits:

$$\begin{aligned} \int_{-1}^1 \frac{dx}{x^{1/3}}\,dx &= \int_{-1}^0 \frac{dx}{x^{1/3}}\,dx + \int_0^1 \frac{dx}{x^{1/3}}\,dx \\ &= \lim_{a\to 0^-} \int_{-1}^a \frac{dx}{x^{1/3}}\,dx + \lim_{b\to 0^+} \int_b^1 \frac{dx}{x^{1/3}}\,dx \\ &= \lim_{a\to 0^-} \Big[\tfrac{3}{2}x^{2/3} \Big]_{-1}^a + \lim_{b\to 0^+} \Big[\tfrac{3}{2}x^{2/3} \Big]_b^1 \\ &= \Big[0 - \tfrac{3}{2}(-1)^{2/3} \Big] + \Big[\tfrac{3}{2}(1)^{2/3} - 0 \Big] \\ &= -\frac{3}{2} + \frac{3}{2} = \boxed{0} \end{aligned}$$

Sketch the graph and you will see why the result should be 0, by symmetry.

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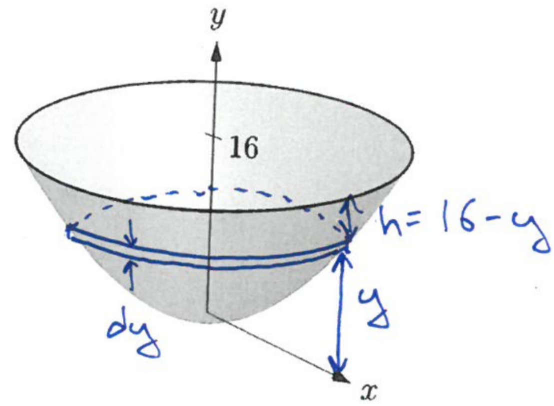
Problem 5: A 16 m deep diving pool is made by revolving the graph of $y = x^2$ $(0 \le x \le 4)$ about the y-axis. The pool is full of water. How much work does is take to pump out all the water to the level of the top of the tank? Express your answer in terms of the density of water, ρ, and the acceleration of gravity, g.



Each slice is a circle of mass

$$m = (\text{density})(\text{volume}) = \rho\pi r^2\, dy = \rho\pi x^2\, dy = \rho\pi y\, dy$$

Moving each thin slice to the top of the pool requires work

$$dW = mgh = (\rho\pi y\, dy)g(16 - y) = \pi\rho g y(16 - y)\, dy$$

The total work is then

$$\begin{aligned} W = \int dW &= \int_0^{16} \pi\rho g y(16-y)\, dy \\ &= \pi\rho g \int_0^{16} (16y - y^2)\, dy \\ &= \pi\rho g \left[8y^2 - \tfrac{1}{3}y^3 \right]_0^{16} \\ &= \pi\rho g \left[8 \cdot 16^2 - \tfrac{1}{3}16^3 \right] = \boxed{\pi\rho g \frac{2048}{3}} \end{aligned}$$