THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

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MIDTERM EXAM #2
SOLUTIONS

17 Mar 2016 10:00–11:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 4 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		9
2		6
3		4
4		4
5		6
TOTAL:		29

/9

Problem 1: Evaluate the following:

/3 (a) $\displaystyle\int_0^\infty \frac{4x}{x^2+1}\,dx$

with $u = x^2+1$ we have

$$\int \frac{4x}{x^2+1}\,dx = \int \frac{2\,du}{u} = 2\ln|u| = 2\ln(x^2+1)$$

so for this improper integral:

$$\begin{aligned}\int_0^\infty \frac{4x}{x^2+1}\,dx &= \lim_{b\to\infty}\int_0^b \frac{4x}{x^2+1}\,dx \\ &= \lim_{b\to\infty}\Big[2\ln(x^2+1)\Big]_0^b \\ &= \lim_{b\to\infty}[2\ln(b^2+1)-0] = \boxed{\infty \quad \text{(diverges)}}\end{aligned}$$

/3 (b) $\displaystyle\int_0^1 \frac{\ln x}{x}\,dx$

With $u = \ln x$ we have

$$\int \frac{\ln x}{x} = \int u\,du = \frac{1}{2}u^2 = \frac{1}{2}(\ln x)^2$$

so for this improper integral:

$$\begin{aligned}\int_0^1 \frac{\ln x}{x}\,dx &= \lim_{b\to 0^+}\int_b^1 \frac{\ln x}{x}\,dx \\ &= \lim_{b\to 0^+}\left[\frac{1}{2}(\ln x)^2\right]_b^1 \\ &= \lim_{b\to 0^+}[0-\frac{1}{2}(\ln b)^2] = \boxed{-\infty \quad \text{(diverges)}}\end{aligned}$$

/3 (c) $\displaystyle\int_0^\pi \cos^3 x\sin^6 x\,dx$

$$\begin{aligned}\int_0^\pi \cos^3 x\sin^6 x\,dx &= \int_0^\pi \cos^2 x\sin^6 x\cos x\,dx \\ &= \int_0^\pi (1-\sin^2 x)\sin^6 x\cos x\,dx\end{aligned}$$

With $u = \sin x$:

$$\int_0^\pi (1-\sin^2 x)\sin^6 x\cos x\,dx = \int_0^0 (1-u^2)u^6\,du = \boxed{0}$$

/6

Problem 2: Evaluate the following:

/3 (a) $\displaystyle\int \frac{2x-5}{x^2+10x+16}\,dx$

Partial fractions:

$$\frac{2x-5}{(x+2)(x+8)} = \frac{A}{x+2} + \frac{B}{x+8} = \frac{A(x+8)+B(x+2)}{(x+2)(x+8)}$$

Matching numerators gives:

$$x=-2: \quad -9 = 6A \quad \Longrightarrow A = -3/2$$

$$x=-8: \quad -21 = -6B \quad \Longrightarrow B = 7/2$$

so we have

$$\int \frac{2x-5}{x^2+10x+16}\,dx = \int \frac{-3/2}{x+2}\,dx + \int \frac{7/2}{x+8}\,dx$$

$$= \boxed{-\frac{3}{2}\ln|x+2| + \frac{7}{2}\ln|x+8| + C}$$

/3 (b) $\displaystyle\int \frac{x^2-3x+4}{x+5}\,dx$

Polynomial long division (or even synthetic division) gives

$$\frac{x^2-3x+4}{x+5} = x - 8 + \frac{44}{x+5}$$

so that

$$\int \frac{x^2-3x+4}{x+5}\,dx = \int x - 8 + \frac{44}{x+5}\,dx$$

$$= \boxed{\frac{1}{2}x^2 - 8x + 44\ln|x+5| + C}$$

/4

Problem 3: For each of the following functions $f(x)$ evaluate the derivative $f'(x)$:

/2 (a) $f(x) = \displaystyle\int_0^x \cos^2 t\,dt$

The fundamental theorem of calculus ("derivatives and integrals are inverses") gives:

$$\boxed{f'(x) = \cos^2 x}$$

/2 (b) $f(x) = \displaystyle\int_x^{\sin x} \frac{1}{\cos z}\,dz$

The fundamental theorem together with the chain rule gives:

$$\boxed{f'(x) = \frac{1}{\cos(\sin x)}\cdot\cos x - \frac{1}{\cos x}}$$

/4 **Problem 4:** The integral $\int_0^1 e^{\sin x}\,dx$ is very difficult to evaluate—I don't know how to do it, and even Wolfram Alpha fails to give an exact answer! Find an approximate value for this integral using Simpson's Rule with $n = 6$ sub-intervals.

$n = 6$ subintervals on $[0,1]$ gives $\Delta x = \frac{1}{6}$ and equally spaced points at:

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6} = \frac{2}{3}, \quad x_3 = \frac{3}{6} = \frac{1}{2}$$
$$x_4 = \frac{4}{6} = \frac{2}{3}, \quad x_5 = \frac{5}{6}, \quad x_6 = \frac{6}{6} = 1$$

Simpson's rule gives

$$\int_0^1 f(x)\,dx \approx \frac{\Delta x}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)\big]$$

$$= \frac{1/6}{3}\big[e^{\sin 0} + 4e^{\sin\frac{1}{6}} + 2e^{\sin\frac{1}{3}} + 4e^{\sin\frac{1}{2}} + 2e^{\sin\frac{2}{3}} + 4e^{\sin\frac{5}{6}} + e^{\sin 1}\big] \approx \boxed{1.63}$$

/6 **Problem 5:** A spherical tank, radius 8 m, is full of water (density $\rho = 1000\,\mathrm{kg/m^3}$). Calculate how much work is required to pump out all the water through a hole at the top of the tank.

Each thin horizontal slice of water is a circle, radius x. Let y be the vertical height of the slice above the center of the sphere. Then $x^2 + y^2 = 8^2$. The work to remove each slice is

$$\begin{aligned} dW &= mgh \\ &= [\rho\pi x^2 dy]g(8-y) \\ &= \rho\pi(8^2 - y^2)g(8-y)\,dy. \end{aligned}$$

$$\begin{aligned} \implies W = \int dW &= \pi\rho g \int_{-8}^{8} (64 - y^2)(8-y)\,dy \\ &= \pi\rho g \int_{-8}^{8} (y^3 - 8y^2 - 64y + 512)\,dy \\ &= 2\pi\rho g \int_0^8 (-8y^2 + 512)\,dy \quad \text{(by symmetry)} \\ &= 2\pi\rho g \Big[-\frac{8}{3}y^3 + 512y\Big]_0^8 \\ &= \boxed{\frac{16384}{3}\pi\rho g \approx 53.5 \times 10^6\ \mathrm{J}} \end{aligned}$$