THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

27 March 2014 10:00–11:15

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.

PROBLEM	GRADE	OUT OF
1		12
2		6

/12 **Problem 1:** Evaluate the following:

(Note the following trigonometric identities: $\sin^2 x + \cos^2 x = 1$, $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$)

/4 (a) $\displaystyle\int_1^3 \frac{dx}{\sqrt{3-x}}$

$$\begin{aligned}
&= \lim_{t\to 3^-} \int_1^t \frac{dx}{\sqrt{3-x}} \\
&= \lim_{t\to 3^-} \left[-2(3-x)^{1/2}\right]_1^t \\
&= \lim_{t\to 3^-} \Big[\underbrace{-2(3-t)^{1/2}}_{\to 0} + 2(3-1)^{1/2}\Big] \\
&= \boxed{2\sqrt{2}}
\end{aligned}$$

/4 (b) $\displaystyle\int_0^\pi \cos^{10} x \sin^3 x \, dx$

$$\begin{aligned}
&= \int_0^\pi \cos^{10} x \sin^2 x \; \sin x \, dx \\
&= \int_0^\pi \cos^{10} x (1-\cos^2 x) \; \sin x \, dx \\
&= \int_1^{-1} -u^{10}(1-u^2)\, du \qquad (u = \cos x) \\
&= \int_{-1}^1 (u^{12} - u^{10})\, du \\
&= \left[\frac{1}{13}u^{13} - \frac{1}{11}u^{11}\right]_{-1}^1 \\
&= 2\left(\frac{1}{11} - \frac{1}{13}\right) = \boxed{\frac{4}{143}}
\end{aligned}$$

/4 (c) $\displaystyle\int \frac{x^2+2}{(x-1)(2x-8)(x+2)}\, dx$

Use partial fractions:

$$\frac{x^2+2}{(x-1)(2x-8)(x+2)} = \frac{A}{x-1} + \frac{B}{2x-8} + \frac{C}{x+2}$$

$$\implies A(2x-8)(x+2) + B(x-1)(x+2) + C(x-1)(2x-8) = x^2 + 2$$

/6

Problem 2: Solve the differential equation

$$\frac{dy}{dt} = ye^{-t}$$

with the "initial condition" $y(0) = 1$.

Solve by separating variables then integrating:

$$\frac{dy}{y} = e^{-t}\,dt \implies \ln|y| = -e^{-t} + C$$

$$\implies |y| = e^{-e^{-t}+C} = Ae^{-e^{-t}} \qquad (A = e^C)$$

$$\implies y = Ae^{-e^{-t}} \qquad (A \text{ can be positive or negative})$$

then impose initial conditions:

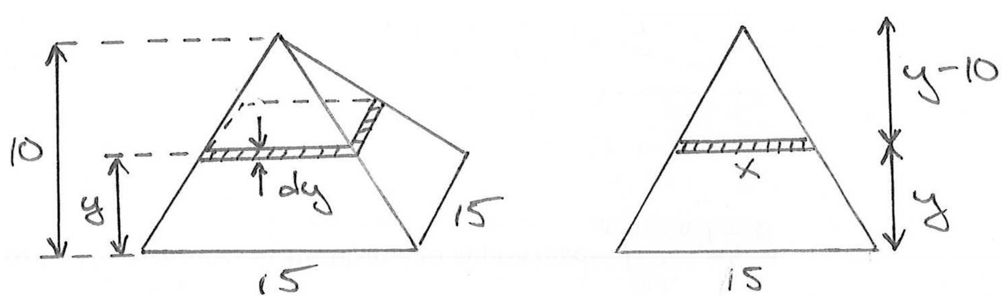
$$1 = y(0) = Ae^{-1} \implies A = e$$

$$\implies \boxed{y = ee^{-e^{-t}} = e^{(1-e^{-t})}}$$

/6

Problem 3: A pyramid of height 10 m, with square base of side length 15 m, is built of stone with density $\rho = 2000\,\text{kg/m}^3$. Calculate the work against gravity required to build this pyramid? (Use $g = 9.8\,\text{m/s}^2$.)

We can construct the pyramid but stacking thin horizontal slices (squares) each of thickness dy, each of which must be raised to a certain height y from the ground.



Let x be the width of a given square slice. Similar triangles gives

$$\frac{x}{10-y} = \frac{15}{10} \implies x = \frac{3}{2}(10-y)$$

so that each slice has mass

$$dm = \rho x^2\,dy = \rho\left[\frac{3}{2}(10-y)^2\right]\,dy.$$

Lifting each slice to its final height y therefore requires work

$$dW = (dm)gy = \rho g y\left[\frac{3}{2}(10-y)^2\right]\,dy.$$

The total work is therefore

/11

Problem 4: (a) Show that the Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

/3

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \\ &\vdots \end{aligned} \implies \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \\ &\vdots \\ f^{(2n)}(0) &= (-1)^n \end{aligned}$$

The Taylor/Maclaurin series formula then gives

$$\begin{aligned} \cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n &= 1 + 0 \cdot x - \frac{1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \end{aligned}$$

(b) For what values of x does the series in part (a) converge?

/3

Apply the ratio test with $a_n = \dfrac{(-1)^n}{(2n)!}x^{2n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{x^{2(n+1)}/(2[n+1])!}{x^{2n}/(2n)!} = \frac{x^{2n+2}}{x^{2n}} \cdot \frac{(2n)!}{(2n+2)!} = x^2 \frac{(2n)!}{(2n)!(2n+1)(2n+2)} = \frac{x^2}{(2n+1)(2n+2)}$$

$$\implies L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)} = 0 < 1$$

so the series converges for *all* $x \in \mathbb{R}$.

(c) Find the Maclaurin series for $f(x) = \cos(x^2)$.

/2

$$\begin{aligned} \cos(x^2) &= 1 - \frac{(x^2)^2}{2!} + \frac{(x^2)^4}{4!} - \frac{(x^2)^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n}}{(2n)!} \\ &= \boxed{1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!}} \end{aligned}$$

(d) Use your answer to part (b) to find a series representation of

$$\int_0^{1/2} \cos(x^2)\, dx$$

and use your series to find an approximate value for this integral, accurate to 5 decimal places.

/3

/9 **Problem 5:** Consider the function $f(x) = \begin{cases} Cx(2-x), & 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$

(a) For what value of C is $f(x)$ a probability density function?

/3

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^2 Cx(2-x)\,dx = C\int_0^2 (2x - x^2)\,dx$$
$$= C\left[x^2 - \frac{1}{3}x^3\right]_0^2 = C\cdot\frac{4}{3} \implies \boxed{C = \frac{3}{4}}$$

(b) Suppose X is a random variable with probability density $f(x)$ as above. Calculate the probability that $X < \frac{1}{4}$.

/3

$$\text{prob}\left(X < \frac{1}{4}\right) = \int_{-\infty}^{1/4} f(x)\,dx = \int_0^{1/4} \frac{3}{4}x(2-x)\,dx$$
$$= \frac{3}{4}\left[x^2 - \frac{1}{3}x^3\right]_0^{1/4} = \boxed{\frac{11}{256} \approx 0.043}$$

(c) Suppose X is a random variable with probability density $f(x)$ as above. Calculate its mean μ.

/3

$$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^2 x\cdot\frac{3}{4}x(2-x)\,dx$$
$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$