THOMPSON RIVERS UNIVERSITY

MATH 1240
Calculus II

Instructor: Richard Taylor

MIDTERM EXAM #2
SOLUTIONS

17 November 2016   11:30–12:45

Instructions:

1. Read the whole exam before beginning.
2. Make sure you have all 5 pages.
3. Organization and neatness count.
4. Justify your answers.
5. Clearly show your work.
6. You may use the backs of pages for calculations.
7. You may use an approved calculator.

PROBLEM	GRADE	OUT OF
1		12
2		6
3		3
4		5
5		3
6		5
7		5
TOTAL:		39

/12 **Problem 1:** Evaluate the following integrals. These identities might help:

$$\sin^2 x + \cos^2 x = 1 \qquad \cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

/3 (a) $\displaystyle\int \frac{1}{x^2-9}\,dx$

Partial fractions:

$$\frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3} = \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$$

$$x=-3: \quad -6A = 1 \quad \Longrightarrow A = -\frac{1}{6}$$
$$x=3: \quad 6B = 1 \quad \Longrightarrow B = \frac{1}{6}$$

so that

$$\int \frac{1}{x^2-9}\,dx = \int \frac{-1/6}{x+3} + \frac{1/6}{x-3}\,dx$$
$$= \boxed{-\tfrac{1}{6}\ln|x+3| + \tfrac{1}{6}\ln|x-3| + C}$$

/3 (b) $\displaystyle\int \frac{x^2+2}{x+1}\,dx$

Polynomial long division (or synthetic division) gives

$$\frac{x^2+2}{x+1} = x - 1 + \frac{3}{x+1}$$

so that

$$\int \frac{x^2+2}{x+1}\,dx = \int x - 1 + \frac{3}{x+1}\,dx$$
$$= \boxed{\tfrac{1}{2}x^2 - x + 3\ln|x+1| + C}$$

/3 (c) $\displaystyle\int \sin^3 x \cos^2 x\,dx$

Rewriting this as

$$\int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \underbrace{\sin x\,dx}_{-du}$$

suggests the substitution $u = \cos x,\ du = -\sin x\,dx$:

$$\int (1-\cos^2 x)\cos^2 x \sin x\,dx = -\int (1-u^2)u^2\,du$$
$$= \int u^4 - u^2\,du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C$$
$$= \boxed{\tfrac{1}{5}\cos^5 x - \tfrac{1}{3}\cos^3 x + C}$$

/3 (d) $\displaystyle\int \sin^2 x\,dx$

Using a double-angle identity:

$$\int \sin^2 x\,dx = \int (\tfrac{1}{2} - \tfrac{1}{2}\cos 2x)\,dx$$
$$= \boxed{\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C}$$

/6 **Problem 2:** Evaluate the following improper integrals:

/3 (a) $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{b\to\infty}\Big[\arctan x\Big]_0^b\\ &= \lim_{b\to\infty}\big[\arctan b - \arctan 0\big]\\ &= \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}\end{aligned}$$

/3 (b) $\displaystyle\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$

The integrand is discontinuous at $x = 0$ so

$$\begin{aligned}\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx &= \lim_{a\to 0}\int_a^8 x^{-1/3}\,dx\\ &= \lim_{a\to 0}\Big[\tfrac{3}{2}x^{2/3}\Big]_a^8\\ &= \lim_{a\to 0}\Big[\tfrac{3}{2}\cdot 8^{2/3} - \tfrac{3}{2}a^{2/3}\Big]\\ &= \tfrac{3}{2}\cdot 8^{2/3} - \tfrac{3}{2}\cdot 0^{2/3}\\ &= \tfrac{3}{2}\cdot 4 = \boxed{6}\end{aligned}$$

/3 **Problem 3:** Let $f(x) = \displaystyle\int_{x^2}^{10} \frac{1}{z^3+1}\,dz$. Evaluate $f'(x)$.

We can write

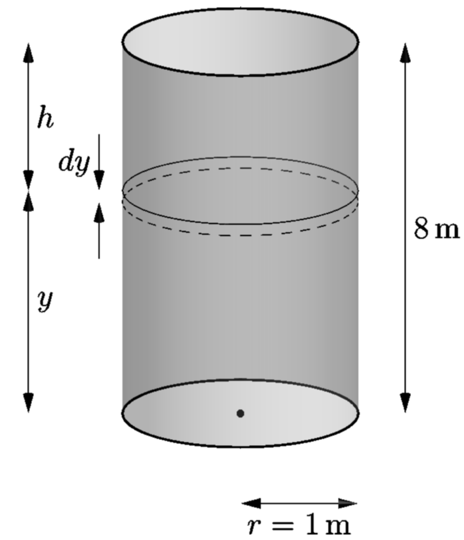
$$f(x) = -\int_{10}^{x^2} \frac{1}{z^3+1}\,dz$$

so by the Fundamental Theorem of Calculus (with the chain rule):

$$f'(x) = -\frac{1}{(x^3)^3+1}\cdot\frac{d}{dx}x^2 = \boxed{-\frac{2x}{x^6+1}}$$

/5

Problem 4: A cylindrical tank, 8 m tall and with diameter 2 m, is full of oil of density ρ [kg/m^3]. How much work is required to pump out *half* of the oil through a hole in the top of the tank? Express your answer in terms of ρ and g (the acceleration of gravity).



To raise the thin circle (thickness dy) shown to the top of the tank requires work

$$dW = mgh$$

where

$$m = \rho \cdot \pi r^2\,dy = \rho\pi(1)^2\,dy = \pi\rho\,dy$$
$$h = 8 - y$$

$$\Longrightarrow dW = \pi\rho g(8-y)\,dy.$$

So the total work to drain half the tank is

$$W = \int dW = \int_4^8 \pi\rho g(8-y)\,dy$$
$$= \pi\rho g\left[8y - \tfrac{1}{2}y^2\right]_4^8$$
$$= \pi\rho g\left[(8\cdot 8 - \tfrac{1}{2}(8)^2) - (8\cdot 4 - \tfrac{1}{2}(4)^2)\right] = \boxed{8\pi\rho g}$$

/3

Problem 5: Calculate the average value of $\cos x$ on the interval $[0, a]$. Express your answer in terms of a.

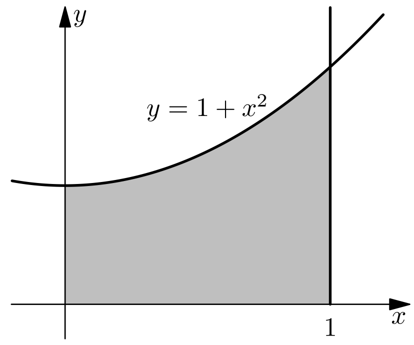
The average value is

$$\bar{f} = \frac{1}{a}\int_0^a \cos x\,dx = \frac{1}{a}\sin x\Big|_0^a = \boxed{\frac{\sin a}{a}}$$

/5 **Problem 6:** The region bounded by the curves

$$y = x^2 + 1, \quad y = 0, \quad x = 0, \quad x = 1$$

is revolved about the y-axis. Calculate the volume of the resulting solid of revolution.



A thin vertical strip of width dx, after revolution about the x-axis, contributes volume $dV = 2\pi r h\, dx$ where

$$h = 1 + x^2, \qquad r = x \qquad \Longrightarrow \qquad dV = 2\pi x(1+x^2)\, dx.$$

The total volume is therefore

$$\begin{aligned} V = \int dV &= \int_0^1 2\pi x(1+x^2)\, dx \\ &= 2\pi \int_0^1 (x + x^3)\, dx \\ &= 2\pi \left[\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 \right]_0^1 = 2\pi \left[\tfrac{1}{2} + \tfrac{1}{4} \right] = \boxed{\frac{3\pi}{2}} \end{aligned}$$

/5 **Problem 7:** Consider the graph of $y = x^3$ on the interval $[0, 1]$.

/2 (a) Write a definite integral that represents the length of this curve.

Since $y' = 3x^2$ we have

$$L = \int_0^1 \sqrt{1 + (y')^2}\, dy = \boxed{\int_0^1 \sqrt{1 + 9x^4}\, dx}$$

/3 (b) Use Simpson's rule (with $n = 4$) to approximate the value of the integral from part (a).

Simpson's rule on $[0, 1]$ with $n = 4$ gives

$$h = 0.25, \quad x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1.$$

Let $f(x) = \sqrt{1 + 9x^4}$. Then

$$\begin{aligned} L = \int_0^1 f(x)\, dx &\approx \frac{h}{3}\big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \big] \\ &= \frac{0.25}{3}\big[\sqrt{1 + 9 \cdot 0^4} + 4\sqrt{1 + 9(0.25)^4} + 2\sqrt{1 + 9(0.5)^4} + 4\sqrt{1 + 9(0.75)^4} + \sqrt{1 + 9 \cdot 1^4} \\ &\approx \boxed{1.548} \end{aligned}$$

Wolfram Alpha gives the exact answer $1.54787\cdots$. Simpson's rule gives a very accurate approximation in this case.